

SUBNORMAL STRUCTURE IN INFINITE SOLUBLE GROUPS

by

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Firstly I extend my warmest thanks to my supervisor, Dr David McCaughan, for his tolerant and constructive guidance during the course of this research.

STATEMENT

The work contained in the body of this thesis is my own except where otherwise indicated.

As regards the attached paper [A]: Sections 1 and 2 are preliminary; Lemmas 3.1-3.5 are due to me; Lemmas 3.6 and 3.7, and the corollary to Theorem B, are due to my co-author; the remainder is joint work not attributable to either author alone.

Finally I am grateful to the Australian National University for the award of a research scholarship.

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SUMMARY

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After an introductory chapter, the definitions and properties of the various classes are treated in Chapter 2. In Chapter 3 we present some technical results to be used in later investigations. Chapter 4 deals with the topic of "rank" in soluble groups and leads up to the result that an extension of a group with the minimal condition on subnormal subgroups extended by a group with bounded subnormal indices again has bounded subnormal indices.

In Chapter 5 we consider, mainly, metanilpotent groups with the subnormal intersection property. Various conditions are given under which such groups are nilpotent, and in the simpler cases the general structure is elucidated. As a first application of these results we prove that an abelian-by-finite group with the subnormal intersection property has a bound on its subnormal indices. The same is true of a

SUMMARY

The basic concepts of this thesis are those of subnormal subgroup and subnormal index. Our aim is to investigate the properties of groups in some classes defined by subnormality conditions, for example the class of groups in which the subnormal indices are bounded. Associated with this are two larger and distinct classes, the first consisting of those groups which have the subnormal intersection property, that is, in which the intersection of any family of subnormal subgroups is again subnormal, the second consisting of those groups which have the subnormal join property, defined analogously.

We attempt to answer two general questions.

- (i) Under what restrictions will a soluble group with some subnormality condition be nilpotent?
- (ii) Under what restrictions will a soluble group with the subnormal intersection property have a bound on its subnormal indices?

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In Chapter 5 we consider, mainly, metanilpotent groups with the subnormal intersection property. Various conditions are given under which such groups are nilpotent, and in the simpler cases the general structure is elucidated. As a first application of these results we prove that an abelian-by-finite group with the subnormal intersection property has a bound on its subnormal indices. The same is true of a

nilpotent-by-(finite nilpotent) group, but the obvious common generalisation is elusive.

In Chapter 6, using a seemingly new restriction, two further results of type (i) are proved. The first of these is used to show that a soluble minimax group with the subnormal intersection property has a bound on its subnormal indices. An example is constructed to show that the same is not true for soluble groups of finite reduced rank.

Also included in the thesis is the joint paper [A] which applies some earlier results to investigate and characterise nilpotent-by-(periodic nilpotent) groups with bounded subnormal indices. We show (e.g.) that a reduced periodic group is of this type if and only if its Sylow subgroups are all nilpotent of bounded class. Similar but weaker results hold for a group which has both the subnormal join and the subnormal intersection property. An example is given to show the incompleteness of the results in the second case.

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CHAPTER 1

INTRODUCTION

It would be a fair statement to say that the central theme of this thesis originates from the concept of "nachinvariant" subgroup introduced by Wielandt in 1939 ([31]). His generalisation of the normality of a subgroup was later termed "accessibility" and finally "subnormality", the latter being standard usage today. At first the development of Wielandt's idea was confined principally to the theory of finite groups, but in the last ten years, mainly due to the efforts of Robinson and Roseblade, its applications to the study of infinite groups have been explored to some extent.

From this standpoint, a group is considered in terms of its "subnormal structure", a term which includes such questions as how many subnormal subgroups there are, where they fit into the lattice of all subgroups of the group, and whether or not they are well-behaved in some sense or other. Investigations from this point of view have generally been directed towards soluble groups, for in this respect simple groups live up to their name by having trivial subnormal structure. At the opposite end of the spectrum are nilpotent groups, for in these each subgroup is subnormal. The interplay between the subnormal structure and the general structure of groups lying between these two extremes has received attention in recent years. In particular, interest seems to have centred on conditions on group structure, both subnormal and general, which will bring the group "close" to nilpotency in some way. Many of the results of this thesis involve conditions of this kind.

Techniques concerned with the manipulation of commutator subgroups play an important rôle in the theory of subnormal subgroups, and it is with these that we begin chapter 2. The idea of subnormal subgroup is made precise: in particular, Robinson's useful concept of standard series is introduced and developed. The question of "where" a subnormal subgroup lies is embodied in the notion of defect or subnormal index. Although most of the material of this chapter is due to Robinson and can be found in his papers [22] - [28] or in [29], I felt it advisable to include the majority of proofs, not only for their intrinsic interest, but also to facilitate the transition to later chapters, where some of the techniques will be used continually and often without explicit reference.

In later stages of the chapter I introduce some classes of groups which are crucial to the topics discussed in the thesis. The first of these is the class of groups in which the subnormal indices are bounded (so that in this sense the subnormal subgroups are well-behaved). This class is the union of an ascending chain of proper subclasses, the first of which consists of those groups in which every subnormal subgroup is normal. Groups with this property, that is, in which "normality is transitive", have been investigated quite intensely, in the context of both finite groups ([2], [7], [28], [32]) and infinite soluble groups ([22]). More generally, soluble groups which have an upper bound on the defects of their subnormal subgroups have been studied in [21], [24], [26] and the attached paper [A]. Their complexity can be gauged from Theorem D of [26] which states, in effect, that an arbitrary soluble group can be embedded in a soluble group of this type.

In the papers mentioned above it becomes clear that a larger but closely related class is that in which the groups have the subnormal intersection property. This is the other important class discussed in chapter 2. The bulk of this thesis consists of attempts to obtain information about soluble groups in this class, beginning, usually, in very simple situations and then seeking to generalise or extend the results to more complex ones, following the pattern of [24]. Such information is contained in chapters 5 and 6, as well as in the attached paper [A].

In dealing with groups which have the subnormal intersection property, it transpires that the notions of π -radicability and π -torsion-freeness play an important part. Chapter 3, therefore, begins with an outline of some basic facts on these concepts, which fits conveniently into the setting of ZA-groups. These properties have been studied in [3], [5] and [18]. Many of the proofs can be found in [17] or [29], so they are generally omitted. The most important part of Chapter 3, as regards applicability for the purposes of this thesis, is Section 3.5, in which are proved a series of inter-related lemmas. They deal with the action of automorphisms on abelian groups which are restricted in terms of π -radicability or π -torsion-freeness. These simple and easily proved results do not seem to appear anywhere in the literature, perhaps because of their specialised nature. Some of them may have wider applications.

In Chapter 4 I have attempted to give a fairly brief and cohesive account of some aspects of the thorny topic of "rank" in soluble groups. Most of the treatment is standard, following [6] or [29], with occasional excursions in the direction of [19]; proofs are again

generally omitted. The main reason for the inclusion of this chapter (in particular the lemmas of Section 4.4) is to pave the way for Chapter 6, where some knowledge of soluble groups of finite rank is essential background. So also is a familiarity with soluble minimax groups, further information on which may be found in [1] and [27].

Chapter 5 sees the first use of the lemmas of Chapter 3 to prove results linking subnormal structure with nilpotency. An important example of this type of result is Lemma 4 of [24], in which Robinson shows that a group with the subnormal intersection property which is a cyclic extension of a free abelian group of finite rank is nilpotent. Motivated by this result, but more interested in abelian-by-finite groups (for reasons which will become clear in the discussion of chapter 6), I proved the fundamental theorem 5.11, at first in a very complex manner, but then more simply as the essential features become evident. Most of the remainder of chapter 5 investigates simple cases of soluble groups with the subnormal intersection property, from which one can deduce detailed information on more general groups of the same type. Unfortunately the complexity of the general picture means that the task of characterising these groups is likely to prove a difficult one. In spite of this some useful results are obtained.

As well as the standard question of what conditions, in addition to the subnormal intersection property, will suffice to ensure the nilpotency of a soluble group, I was also interested in the (clearly weaker) conditions required to imply the existence of a bound for the subnormal indices. Some results in this direction are also presented in chapter 5. These finally led to the (unstated) theorem that a soluble, abelian-by-finite group with the subnormal intersection

property has a bound for its subnormal indices. With a little trickery, the condition of solubility can be dropped, to yield the final theorem 5.46 of this chapter.

In chapter 6, attention focuses on soluble minimax groups. The reason for this interest lies in the following facts: a soluble group with the minimal condition on subgroups has a bound on its subnormal indices ([25], Lemma 3.2); and a soluble group with the maximal condition on subgroups, being finitely generated, has a bound on its subnormal indices if it has the subnormal intersection property, by a result of Robinson ([24], Theorem A). It is natural to ask, therefore, "For what classes of groups containing all soluble groups with the minimal or maximal conditions on subgroups do the two subnormality properties coincide?" The class of soluble minimax groups springs to mind.

It is at this stage that the reason for my interest in abelian-by-finite or, more generally, nilpotent-by-finite groups becomes clear, for in this context, McDougall ([20], Theorem A) has proved a result showing that the above question, when restricted to soluble minimax groups, reduces to a consideration of the nilpotent-by-finite case. Once again the key theorem (6.12) is one which deals with conditions on a soluble group with the subnormal intersection property which force it to be nilpotent. This result is then utilised to show that a soluble minimax, nilpotent-by-finite group with the subnormal intersection property has a bound for its subnormal indices, and the desired theorem is an easy consequence. An example is given to show that a soluble group of finite reduced rank may have the subnormal intersection property but fail to have a bound on its subnormal indices. I have been unable to decide whether the two properties

coincide for soluble groups of finite total rank. Another interesting, but seemingly difficult question which is left undecided is whether an arbitrary nilpotent-by-finite group with the subnormal intersection property has a bound on its subnormal indices. The indications of chapters 5 and 6 are that it should, but the proof will not be easy.

Part of this thesis is the paper [A], which was written jointly in late 1971 with my supervisor. He realised that some of my results (in particular, the one which appears as Lemma 3.5 of [A]) could be used to investigate metanilpotent groups with bounds on the subnormal indices of their subnormal subgroups, his earlier efforts in this direction having foundered for lack of suitable tools.

The initial sections, 1 and 2, are purely preliminary; indeed section 2 is a resumé of some of the material covered in chapters 2 and 3 of this thesis. Section 3 begins with a sequence of technical lemmas: 3.1 - 3.5 are due to me, and 3.6 - 3.7 to my co-author. Some of these lemmas have been restated, for convenience, in the body of the thesis, with references to [A] for the proofs. With the exception of the corollary to Theorem B, which is due to my co-author, the rest of the results of [A] are joint work not attributable to either of us alone.

The main results of the paper concern nilpotent-by-(periodic nilpotent) groups, but are more readily explained for periodic metanilpotent groups. It is shown that a reduced group in this latter class has a bound on its subnormal indices if and only if its Sylow subgroups are all nilpotent of bounded class. Weaker and less complete versions of this result are proved for groups in which the subnormal subgroups form a complete sub-lattice of the lattice of all

subgroups, that is, groups which have both the subnormal join and the subnormal intersection property. In this case the condition involving the nilpotency of the Sylow subgroups is replaced to some extent by the weaker condition that each of their subgroups should be subnormal. Finally, as is almost traditional in this branch of group theory, an example is constructed to whet the appetites of future researchers (for other branches, perhaps).

2.1 Preliminaries

2.11 DEFINITION. If $\{H_i : 1 \leq i \leq I\}$ is a collection of subsets of a group G , we denote by $\langle H_i : 1 \leq i \leq I \rangle$ the subgroup of G generated by the subsets H_i , $1 \leq i \leq I$, that is, the smallest subgroup of G containing them. (Obvious remarks will be omitted.)

2.12 DEFINITION. (a) If x and y are elements of a group, we denote by x^y the conjugate $y^{-1}xy$, and by $[x, y]$ the commutator

CHAPTER 2

SUBNORMAL SUBGROUPS

The first section of this chapter sets out some basic notation and terminology, mainly with regard to commutators; two simple lemmas are recorded for future use. Then we proceed to the main topic of the chapter - indeed of the thesis - by introducing in Section 2.2 the concept of a subnormal subgroup. After developing some of the elementary theory of subnormal subgroups, we devote the remaining sections of the chapter to a discussion of groups which are restricted in some way by conditions on their subnormal subgroups. The restrictions treated in Sections 2.3 and 2.4 are central to the subject-matter of the thesis. These are, respectively, the requirement that there should be a bound on the defects of subnormal subgroups, and the requirement that the intersection of any family of subnormal subgroups should be subnormal. Section 2.5, which deals with joins of subnormal subgroups, though perhaps of independent interest, is less important and therefore less detailed.

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$x^{-1}y^{-1}xy$. If x_1, x_2, \dots, x_n are elements of a group with $n > 2$, we denote by $[x_1, x_2, \dots, x_n]$ the element $\left[[x_1, \dots, x_{n-1}], x_n \right]$.

(b) If X and Y are subgroups of a group, we denote by X^Y the subgroup $\langle x^y : x \in X, y \in Y \rangle$; we denote by $[X, Y]$ the subgroup $\langle [x, y] : x \in X, y \in Y \rangle$, the commutator of X and Y . If X_1, X_2, \dots, X_n are subgroups of a group, with $n > 2$, we denote by $[X_1, X_2, \dots, X_n]$ the subgroup $\left[[X_1, \dots, X_{n-1}], X_n \right]$.

We state in the form of a lemma some well-known and easily verifiable properties of commutators.

2.13 LEMMA. *If x, y and z are elements of a group then*

$$(i) \quad [x, y] = [y, x]^{-1} ;$$

$$(ii) \quad (a) \quad [x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z] ;$$

$$(b) \quad [xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z] ;$$

$$(iii) \quad [x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1 .$$

We can now deduce some useful and well-known relations between commutator subgroups.

2.14 LEMMA. *If X, Y and Z are subgroups of a group, then*

$$(i) \quad [X, Y] = [Y, X] ;$$

$$(ii) \quad [X, Y]^Y = [X, Y] ;$$

$$(iii) \quad X^Y = \langle X, [X, Y] \rangle = X[X, Y] ;$$

$$(iv) \quad [X, Y, Y] \leq [X, Y] ;$$

$$(v) \quad X^{\langle X, Y \rangle} = X^Y ;$$

$$(vi) \quad \text{if } Y^Z = Y \text{ then } [X, \langle Y, Z \rangle] = \langle [X, Z], [X, Y]^Z \rangle .$$

Proof: (i) and (ii) follow easily from 2.13 (i) and (ii) respectively. (iii) is immediate from the definition, using (i) and (ii) to obtain the second equality. (iv) is a consequence of (ii) and (iii). To prove (v), note that $\langle X, Y \rangle = YX^Y$, so that

$$X^{\langle X, Y \rangle} = X^{YX^Y} = (X^Y)^{X^Y} = X^Y .$$

(vi) is immediate from 2.13 (ii).

The following result, the "three subgroup lemma", is a consequence of 2.13 (iii).

2.15 LEMMA ([8], Theorem 2.3). *If X, Y and Z are normal subgroups of a group G , then*

$$[X, Y, Z] \leq [Y, Z, X][Z, X, Y] .$$

Proof: It is easy to see that each of the commutator subgroups involved is normal in G . It will then suffice to show that $[X, Y, Z]$ is trivial on the assumption that the subgroup $[Y, Z, X][Z, X, Y]$ is trivial. But if the latter is the case, then, by 2.13 (iii), for any $x \in X$, $y \in Y$ and $z \in Z$, $[x, y, z] = 1$. Thus each element of Z commutes with every generator of $[X, Y]$,

showing that $[X, Y, Z] = 1$, as required.

In dealing with commutator subgroups, two situations frequently arise for which it is useful to have some concise notation.

2.16 DEFINITION. (a) If X and Y are subgroups of a group we define γ_{XY}^α for each ordinal $\alpha \geq 0$ by

$$\gamma_{XY}^0 = X ; \quad \gamma_{XY}^\alpha = [\gamma_{XY}^{\alpha-1}, Y]$$

if α is a positive (that is, non-empty) non-limit ordinal;

$$\gamma_{XY}^\alpha = \bigcap_{\beta < \alpha} \gamma_{XY}^\beta$$

if α is a limit ordinal.

(b) If G is a group we define γ_α^G for each positive ordinal α by

$$\gamma_1^G = G ; \quad \gamma_\alpha^G = [\gamma_{\alpha-1}^G, G]$$

if α is a non-limit ordinal, $\alpha > 1$;

$$\gamma_\alpha^G = \bigcap_{\beta < \alpha} \gamma_\beta^G$$

if α is a limit ordinal.

No confusion should arise between these two notations. There is of course a close connection between them: in fact if α is a finite positive ordinal $\gamma_\alpha^G = \gamma_{GG}^{\alpha-1}$, whereas for an infinite ordinal β ,

$$\gamma_\beta^G = \gamma_{GG}^\beta.$$

The *lower central series* of G is the descending chain of fully invariant subgroups of G defined in 2.16 (b). Its second term $\gamma_2 G = [G, G] = G'$ is the *derived group* of G , and the *derived series* of G can be defined by

$$G^{(0)} = G ; \quad G^{(\alpha)} = \gamma_{G^{(\alpha-1)}} G^{(\alpha-1)} = (G^{(\alpha-1)})'$$

if α is a positive, non-limit ordinal;

$$G^{(\alpha)} = \bigcap_{\beta < \alpha} G^{(\beta)}$$

if α is a limit ordinal.

We will assume here the elementary properties of soluble and nilpotent groups, that is, groups in which the trivial subgroup appears after finitely many terms of the derived series or lower central series respectively.

We conclude this section with a useful lemma, a variation on Theorem 2 of [10].

2.2 Subnormal Subgroups

2.17 LEMMA. *If H is a normal subgroup of a group G then for each positive integer k ,*

$$[H, \gamma_k G] \leq \gamma_k H.$$

Proof: We proceed by induction on k , noting firstly that

$$[H, \gamma_1 G] = [H, G] = \gamma_1 H.$$

If $k > 1$, and the lemma holds for integers less than k , we

have

$$\begin{aligned} [H, \gamma_k G] &= [\gamma_k G, H] \\ &= [\gamma_{k-1} G, G, H] , \end{aligned}$$

by definition. Hence, applying Lemma 2.15, we obtain

$$[H, \gamma_k G] \leq [G, H, \gamma_{k-1} G] [H, \gamma_{k-1} G, G] ,$$

since all the subgroups involved are normal in G . But by the induction hypothesis,

$$[G, H], \gamma_{k-1} G \leq \gamma [H, G] G^{k-1} = \gamma H G^k .$$

By the same token,

$$[[H, \gamma_{k-1} G], G] \leq [\gamma H G^{k-1}, G] = \gamma H G^k .$$

It follows that $[H, \gamma_k G] \leq \gamma H G^k$, which completes the proof.

2.2 Subnormal Subgroups

The basic idea of this section is due to Wielandt ([31]); the treatment largely follows that of Robinson [29].

2.21 DEFINITION. A subgroup H of a group G is said to be *subnormal* in G when there is a chain of subgroups

$$H = H_r \leq H_{r-1} \leq \dots \leq H_0 = G$$

for some non-negative integer r , with H_i normal in H_{i-1} for each

i such that $0 < i \leq r$. If such a chain exists, then there will be one of minimal length, that is, a chain in which the number of non-trivial factors H_{i-1}/H_i is least. This number is clearly independent of the choice of a minimal such chain: it is called the *subnormal index* or *defect* of H in G , and denoted by $s(G : H)$.

The following remarks are trivial consequences of the definition.

(a) If H is a subnormal subgroup of G and K is any subgroup of G , then $H \cap K$ is subnormal in K , and $s(K : H \cap K) \leq s(G : H)$.

(b) If K is a subnormal subgroup of G , and H is a subnormal subgroup of K , then H is subnormal in G and $s(G : H) \leq s(G : K) + s(K : H)$.

(c) If H is a subnormal subgroup of G , and N is a normal subgroup of G , then HN is subnormal in G , HN/N is subnormal in G/N , and

$$s(G : HN) = s(G/N : HN/N) \leq s(G : H).$$

Before proceeding to a more detailed discussion of subnormal subgroups, we digress to prove a lemma which, as well as being of use in later chapters, serves to show the similarity, in some circumstances, of the behaviour of normal and subnormal subgroups. It is a generalisation, due to Robinson, of a well-known theorem of Fitting.

2.22 LEMMA ([23], Lemma 4.5). Let $G = HN$, where H is a nilpotent subnormal subgroup of G , and N is a nilpotent normal

subgroup of G . Then G is nilpotent.

Proof: Let $H = H_r \leq H_{r-1} \leq \dots \leq H_0 = G$ be a chain of subgroups as in 2.21. Then H_r is certainly nilpotent. We show that, for $0 < i \leq r$, the nilpotency of H_i implies that of H_{i-1} . Since $H_0 = G$ this will suffice to prove the lemma.

Suppose that H_i is nilpotent. Since $H_i N = G$,

$$H_{i-1} = H_{i-1} \cap H_i N = H_i (H_{i-1} \cap N),$$

by the modular law.

Both H_i and $H_{i-1} \cap N$ are nilpotent normal subgroups of H_{i-1} , so that, by Fitting's theorem, their product H_{i-1} is nilpotent, and the proof is complete.

Now we return to the main theme to develop an alternative approach to subnormal subgroups.

2.23 DEFINITION. If H is a subgroup of a group G , the *standard series* of H in G is a descending chain of subgroups of G , each containing H , defined for each ordinal α by

$$H^{G,0} = G; \quad H^{G,\alpha} = H^{H^{G,\alpha-1}}$$

if α is a positive non-limit ordinal;

$$H^{G,\alpha} = \bigcap_{\beta < \alpha} H^{G,\beta}$$

if α is a limit ordinal.

We make some remarks on this definition.

(a) For any given group G , there is an ordinal λ such that for any subgroup H of G , $H^{G,\lambda} = H^{G,\lambda+1}$. It is then clear that for any ordinal $\alpha \geq \lambda$, $H^{G,\alpha} = H^{G,\lambda}$, and the standard series of H in G may be taken as

$$\{H^{G,\alpha} : \alpha \leq \lambda\}.$$

(b) It is a simple matter to show that if H and K are any subgroups of a group G , and α is an ordinal, then

$$(H \cap K)^{G,\alpha} \leq H^{G,\alpha} \cap K^{G,\alpha}$$

and

$$(H \cap K)^{K,\alpha} \leq H^{G,\alpha} \cap K.$$

(c) If H is a subgroup and N a normal subgroup of G , then for any finite ordinal α ,

$$(HN)^{G,\alpha} = H^{G,\alpha}N$$

and

$$\overline{H}^{\overline{G},\alpha} = H^{G,\alpha}N/N,$$

where $\overline{G} = G/N$ and $\overline{H} = HN/N$.

We now give another version of the standard series of a subgroup

(at least for the "early" terms), using the commutator notation introduced in 2.16.

2.24 LEMMA. *Let H be a subgroup of G . Then for any non-negative integer i ,*

$$H^{G,i} = H\gamma GH^i.$$

Proof: First we remark that for each i , γGH^i is normalised by H , using 2.14 (ii), so that the right hand side of the equality is indeed a subgroup. We prove the statement by induction on i , noting that $H^{G,0} = G = H\gamma GH^0$ by definition. If i is a positive integer, and the statement is true for all non-negative integers less than i , then we have

$$\begin{aligned} H^{G,i} &= H^{H^{G,i-1}} \\ &= H[H^{G,i-1}, H] \quad \text{by 2.14 (iii)} \\ &= H[H\gamma GH^{i-1}, H] \quad \text{by inductive hypothesis} \\ &= H[\gamma GH^{i-1}, H] \quad \text{by 2.14 (vi)}. \end{aligned}$$

Hence $H^{G,i} = H\gamma GH^i$, which completes the proof.

We now prove a lemma linking the idea of subnormality with both versions of the standard series.

2.25 LEMMA. *The following three conditions on a subgroup H of a group G are equivalent:*

(i) H is subnormal in G , with $s(G : H) \leq r$;

$$(ii) \quad H = H^{G,r};$$

$$(iii) \quad \gamma GH^r \leq H.$$

Proof: Suppose (i) is true. Then there is a chain

$$H = H_r \leq H_{r-1} \leq \dots \leq H_0 = G$$

of subgroups, each normal in the next. We prove by induction that for

each i with $0 \leq i \leq r$, $H^{G,i} \leq H_i$, noting that $H^{G,0} = G = H_0$.

Suppose that $0 < i \leq r$ and that $H^{G,i-1} \leq H_{i-1}$. Then

$$H^{G,i} = H^{H^{G,i-1},i} \leq H^{H^{G,i-1},i-1} \leq H_i^{i-1} = H_i.$$

The inductive proof is complete, and $H^{G,r} = H$ follows on taking $i = r$, proving that (i) implies (ii).

The equivalence of (ii) and (iii) is immediate from Lemma 2.24.

Now suppose that (ii) is true. By definition 2.23, each term $H^{G,i}$ of the standard series of H is normal in its predecessor $H^{G,i-1}$, for each positive integer i . The chain

$$H = H^{G,r} \leq H^{G,r-1} \leq \dots \leq H^{G,0} = G$$

is then precisely of the kind described in 2.21, and has at most r non-trivial factors. Thus H is subnormal in G with $s(G : H) \leq r$, establishing (i) and completing the proof of the lemma.

REMARK. It is easy to deduce from the proof of Lemma 2.25 that

if $H^{G,r} < H^{G,r-1}$ for some positive integer r , then $H^{G,r}$ is subnormal in G with defect precisely r .

2.3 Bounds on subnormal indices

In this section we consider some basic properties of the class of groups in which the subnormal indices of subnormal subgroups are bounded. By a *class* of groups we mean a class in the usual sense which contains a trivial group, and with every group in the class, all its isomorphic copies. The class mentioned may be considered as the union of an ascending chain of proper subclasses, one for each non-negative integer n . The subclass corresponding to n is composed of those groups in which the defects of the subnormal subgroups do not exceed n . For example, when $n = 0$ the subclass contains only trivial groups, and when $n = 1$ each group in the subclass has the property that any subnormal subgroup is normal - "normality is transitive".

Our first lemma characterises groups in any of these subclasses in terms of the standard series of an arbitrary subgroup.

2.31 LEMMA ([24], Lemma 2 (ii)). *A group G has a bound n on its subnormal indices if and only if, for each subgroup H of G , $H^{G,n} = H^{G,n+r}$, where r is any non-negative integer.*

Proof: If G has a subgroup H such that $H^{G,n} > H^{G,n+r}$, for some positive integer r , then it is clear from the definition of standard series that $H^{G,n} > H^{G,n+1}$. But by the remark following Lemma 2.25, this means that $H^{G,n+1}$ is a subnormal subgroup of defect precisely $n + 1$ in G . This establishes one half of the lemma.

On the other hand, if for each subgroup H of G' , $H^{G,n} = H^{G,n+r}$ for any non-negative integer r , this will hold *a fortiori* when H is a subnormal subgroup. Since H coincides with some term of its standard series, by Lemma 2.25, it follows that $H = H^{G,n}$, which, by the same lemma, shows $s(G : H) \leq n$. Now H was an arbitrary subnormal subgroup of G , so the proof of the lemma is complete.

The following proposition is an immediate consequence of remarks (a) and (c) after Definition 2.21.

2.32 PROPOSITION. *Let K be a subnormal subgroup and N a normal subgroup of a group G which has a bound on its subnormal indices. Then K and G/N have the same bound on their subnormal indices.*

(We say that subnormal subgroups and quotient groups "inherit" the property of having a bound on the subnormal indices of their subnormal subgroups.)

In his seminal paper [31], Wielandt was concerned largely with groups having a composition series of finite length (see [16], p. 112). The terms of such a series, if it exists, are all subnormal subgroups of the group; indeed the following lemma, which is an obvious consequence of the Jordan-Hölder-Schreier theorem, shows that the length of such a series gives a bound for the defect of any subnormal subgroup.

2.33 LEMMA. *If G is a group with a composition series of length n , then any strictly descending chain of subnormal subgroups*

of G has at most $n + 1$ members.

We can use this to derive a more general result.

2.34 LEMMA ([24], Lemma 1). *Let G be a group with a normal subgroup N which has a composition series of length n . If G/N has a bound r for its subnormal indices, then G has the bound $r + n$ for its subnormal indices.*

Proof: Let H be any subgroup of G . Because G/N has the bound r for the defects of its subnormal subgroups, we may apply Lemma 2.31 to the standard series of HN/N in G/N . Bearing in mind remark 2.23 (c), we deduce that, for any non-negative integer t ,

$$H^{G,r}N = H^{G,r+t}N.$$

Now the descending chain

$$H^{G,r} \cap N \geq H^{G,r+1} \cap N \geq \dots \geq H^{G,r+n+1} \cap N$$

of subnormal subgroups of N has $n + 2$ members. By Lemma 2.33, therefore, there is an integer k with $r \leq k \leq r+n$ such that

$$H^{G,k} \cap N = H^{G,k+1} \cap N.$$

Since $H^{G,k}N = H^{G,k+1}N$, it follows that

$$H^{G,k} = H^{G,k} \cap H^{G,k}N = H^{G,k} \cap H^{G,k+1}N = H^{G,k+1}(H^{G,k} \cap N) = H^{G,k+1}.$$

This means that, at worst,

$$H^{G,r+n} = H^{G,r+n+1},$$

proving the result in view of Lemma 2.31.

In most applications of this result, N is a finite group. If we transpose the conditions on N and G/N in Lemma 2.34, the conclusion of the lemma does not hold: it is not true that a finite extension of a group with bounded subnormal indices again has bounded subnormal indices.

2.35 EXAMPLE. Consider the infinite dihedral group

$$D = \langle a, b : (ab)^2 = b^2 = 1 \rangle.$$

The chain of subgroups $\{D_n : n \geq 0\}$ defined by

$$D_n = \langle a^{2^n}, b \rangle$$

is the standard series of the subgroup $B = \langle b \rangle$, because $D_0 = D$,

and for each integer $n \geq 0$,

$$\begin{aligned} {}^D B^n &= B[D_n, B] \\ &= \langle b, [a^{2^n}, b] \rangle = \langle b, a^{2^{n+1}} \rangle \\ &= D_{n+1}. \end{aligned}$$

Now for each $n \geq 0$, $D_{n+1} < D_n$; thus by the remark after Lemma 2.25 we have $s(D : D_n) = n$, so that D has unbounded subnormal indices. This is in spite of the fact that D has a normal abelian subgroup $A = \langle a \rangle$ with D/A a 2-cycle.

Many common types of groups, however, do have bounds on their

subnormal indices: in particular any nilpotent group will have this property. For if G is nilpotent of class c and H is any subgroup of G ,

$$\gamma_{GH}^c \leq \gamma_{c+1} G = 1.$$

By 2.25 we see that H is subnormal in G with subnormal index at most c .

Great interest centred for some years on converses to this situation. If a group has every subgroup subnormal, is it necessarily nilpotent? If a group has every subgroup subnormal of bounded defect, is it necessarily nilpotent? The first question was answered in the negative by Heineken and Mohamed, who in 1968 ([14]) constructed, for each prime p , a metabelian p -group which is not nilpotent although every proper subgroup is nilpotent and subnormal.

The second question was given a positive answer by Roseblade in some very deep work in 1965 ([30]). We can state the relevant result as follows:

2.36 THEOREM ([30], Corollary to Theorem 1). *There is a function R on the set of positive integers such that if G is a group in which every subgroup is subnormal of defect at most n , then G is nilpotent of class at most $R(n)$.*

An immediate consequence is the following result, which does not seem to appear in the literature.

2.37 COROLLARY. *If G is a group with a bound on its subnormal*

indices, then the lower central series of G terminates after a finite number of steps; indeed there is a non-negative integer j , depending only on the bound, such that for any non-negative integer k ,

$$\gamma_j G = \gamma_{j+k} G.$$

Proof: Let n be the bound on the defects of subnormal subgroups of G . If $n = 0$ there is nothing to prove. If $n > 0$, choose $j = R(n)$. Then for any $k \geq 0$, $G/\gamma_{j+k} G$ is nilpotent, thus has every subgroup subnormal. But, by Proposition 2.32, the bound n on subnormal indices is inherited by the factor group $G/\gamma_{j+k} G$, which therefore must have nilpotent class at most j , by Theorem 2.36. It follows that $\gamma_j G \leq \gamma_{j+k} G$, and, since the reverse inclusion is trivial, the corollary is proved.

Although 2.37 proves useful later in the thesis, perhaps a more interesting consequence of Roseblade's theorem is the following unpublished result of Robinson.

2.38 THEOREM (Robinson). *The direct product of two groups, each with a bound on its subnormal indices, again has a bound on its subnormal indices.*

Proof: Suppose $G = M \times N$ where M and N have a bound k for the subnormal indices of their subnormal subgroups. Let H be any subnormal subgroup of G . Write

$$M_0 = M \cap H, \quad N_0 = N \cap H, \quad M_1 = M \cap NH, \quad N_1 = N \cap MH.$$

Note that since $M \cap H$ is normalised by N and by H , M_0 is normal in M_1 . Similarly N_0 is normal in N_1 . Note also that

$$M_0 \times N_0 \leq H \leq M_1 \times N_1.$$

Now we prove that, for any positive integer i ,

$$\gamma_i^{M_1} = \gamma_{M_1 H}^{i-1}.$$

We use induction on i , remarking that for $i = 1$ the statement is trivial. Suppose that $i > 1$ and that

$$\gamma_{i-1}^{M_1} = \gamma_{M_1 H}^{i-2}.$$

Then

$$\begin{aligned} \gamma_{M_1 H}^{i-1} &= \left[\gamma_{M_1 H}^{i-2}, H \right] \\ &\leq \left[\gamma_{i-1}^{M_1}, M_1 \times N_1 \right] \quad \text{using the induction hypothesis} \\ &= \left[\gamma_{i-1}^{M_1}, M_1 \right] = \gamma_i^{M_1} \\ &= \left[\gamma_{M_1 H}^{i-2}, M \cap NH \right] \\ &\leq \left[\gamma_{M_1 H}^{i-2}, NH \right] \\ &= \left[\gamma_{M_1 H}^{i-2}, H \right] \quad \text{since } [M_1, N] = 1. \end{aligned}$$

Thus $\gamma_i^{M_1} = \gamma_{M_1 H}^{i-1}$, as required.

Now because H is subnormal in G , there is an integer m such that $\gamma_{M_1 H}^m \leq M_1 \cap H = M_0$; it follows that $\gamma_{m+1}^{M_1} \leq M_0$, and thus

M_1/M_0 is nilpotent. But M_1 is subnormal in M , by remarks (c) and (a) after 2.21, so M_1/M_0 inherits the bound k for its subnormal indices. By Theorem 2.36, the nilpotent class of M_1/M_0 is at most $r = R(k)$, hence $\gamma_{M_1} H^r \leq M_0$.

By an exactly parallel argument, $\gamma_{N_1} H^r \leq N_0$. Now it is clear and that for $i \geq 0$, $\gamma_{(M_1 \times N_1)} H^i$ is the direct product of $\gamma_{M_1} H^i$ and $\gamma_{N_1} H^i$; hence $\gamma_{(M_1 \times N_1)} H^r \leq H$, that is, $s(M_1 \times N_1 : H) \leq r$. But M_1 has defect at most k in M , hence in G , and similarly N_1 has defect at most k in G . We deduce that $s(G : M_1 \times N_1) \leq k$, and finally, by remark (b) after 2.21, that $s(G : H) \leq k + r$. Since this integer is independent of the choice of H , the theorem is proved.

It would be interesting to have an "elementary" proof of this result; the heavy machinery used seems inappropriate.

We can now prove a more general result in the same vein.

2.39 THEOREM. *Let M_1 and M_2 be normal subgroups of G with $G/M_1 M_2$ nilpotent. If G/M_1 and G/M_2 have a bound for their subnormal indices, then so does $G/M_1 \cap M_2$.*

Proof: There is no loss of generality in assuming $M_1 \cap M_2 = 1$. Our task now is to show that G has a bound on its subnormal indices.

From remarks (a) and (b) following Definition 2.21, it is clear

The map θ defined by

$$g\theta = (gM_1, gM_2)$$

is a monomorphism from G into $G/M_1 \times G/M_2$. Write

$$N_1 = \{(m_2M_1, M_2) : m_2 \in M_2\}$$

and

$$N_2 = \{(M_1, m_1M_2) : m_1 \in M_1\}.$$

Then N_1 and N_2 are subgroups of $G\theta$; so, then, is their product

$$N = \{(m_2M_1, m_1M_2) : m_1 \in M_1, m_2 \in M_2\}.$$

Now N is normal in $G/M_1 \times G/M_2$, and the corresponding factor group is isomorphic to $G/M_2M_1 \times G/M_1M_2$, so is nilpotent. It follows that $G\theta$ is a subnormal subgroup of $G/M_1 \times G/M_2$. This latter group, however, has a bound on its subnormal indices, by Theorem 2.38; thus $G\theta$, and its isomorphic copy G , must have a bound for its subnormal indices, as required.

2.4 The Subnormal Intersection Property

In this section we introduce another condition on subnormal subgroups which will prove to be less restrictive than that of Section 2.3.

From remarks (a) and (b) following Definition 2.21, it is clear

that in any group the intersection of two subnormal subgroups is a subnormal subgroup. It is not necessarily the case, however, that the intersection of an arbitrary family of subnormal subgroups is subnormal. For example, in the infinite dihedral group, with the notation of 2.35, the chain of subnormal subgroups $\{D_n : n \geq 0\}$ intersects in the subgroup B . But B cannot be a subnormal subgroup, by Lemma 2.25, since its standard series

$$\left\{ B^{D,n} = D_n : n \geq 0 \right\}$$

does not become stationary.

We say that a group G has the *subnormal intersection property* if the intersection of an arbitrary family of subnormal subgroups of G is again a subnormal subgroup of G . The infinite dihedral group, therefore, fails to have the subnormal intersection property. That it is in some sense typical of such groups will be seen from the following lemma.

2.41 LEMMA ([24], Lemma 2 (i)). *A group G has the subnormal intersection property if and only if the standard series of every subgroup becomes stationary after finitely many terms.*

Proof: Suppose G has the subnormal intersection property. Let H be any subgroup, and let

$$Y = \cap \{H^{G,i} : i \geq 0\}$$

be the intersection of those terms of its standard series indexed by finite ordinals. Then, by assumption, Y is subnormal in G .

Because H is contained in Y , each term of the standard series of H is contained in the corresponding term of the standard series of Y , by remark 2.23 (b). But if d is the defect of Y in G , this means that

$$H^{G,d} \leq Y^{G,d} = Y,$$

hence for any integer $i \geq 0$,

$$H^{G,d} \leq H^{G,d+i}.$$

This proves one half of the lemma, since H was an arbitrary subgroup. Conversely, let

$$H = \cap \{H_i : i \in I\}$$

be the intersection of an arbitrary family of subnormal subgroups of G . Then, by assumption, the standard series of H in G becomes stationary, at \bar{H} , say, after finitely many terms. \bar{H} is a subnormal subgroup, by Lemma 2.25 and the succeeding remark. Now for each i in I , H_i , being a subnormal subgroup, coincides with a term of the standard series of H_i in G , and therefore contains the corresponding term of the standard series of H in G , by remark 2.23 (b). *A fortiori*, each H_i certainly contains \bar{H} . It follows easily that $\bar{H} = H$ and that H is subnormal in G , completing the proof of the lemma.

The next proposition is an easy consequence of remarks 2.21 (b) and (c).

2.42 PROPOSITION. *Let K be a subnormal subgroup and N a normal subgroup of a group G which has the subnormal intersection property. Then K and G/N have the same property.*

It may be of interest to note an alternative characterisation of groups with the subnormal intersection property, which does not seem to appear in the literature.

2.43 LEMMA. *A group G has the subnormal intersection property if and only if for each subgroup H of G the family of all subnormal subgroups of G containing H has a unique minimal member \bar{H} .*

Proof: If G has the subnormal intersection property, we can take \bar{H} to be the intersection of all subnormal subgroups of G containing H . Conversely, if G satisfies the second condition and H is any subgroup of G , the corresponding \bar{H} is clearly contained in every term of the standard series of H in G . But \bar{H} coincides with the k -th term of its own standard series, for some non-negative integer k , hence \bar{H} contains the corresponding term of the standard series of H . Thus \bar{H} is the terminal point of the standard series of H in G , showing, by Lemma 2.41, that G has the subnormal intersection property.

It is clear from a comparison of Lemmas 2.31 and 2.41 that any group with a bound on its subnormal indices will have the subnormal intersection property. That the two properties are not the same is shown by our next example. We discuss it in some detail, not only because of its relevance to our main theme but also because some of the techniques used will be needed in later chapters.

2.44 EXAMPLE ([24], Lemma 2 (iii)). Consider the standard restricted wreath product $G = X \wr Y$ of a p -cycle X and a quasi-cyclic p -group Y , where p is any prime. Denote by B the base group of this wreath product. Let T be the ring of endomorphisms of B ; then for any $t \in T$, $(t-1)^p = t^p - 1$, since B has exponent p . If $b \in B$ and $t \in T$ we denote by $[b, t]$ the element $b^{t-1} = b^{-1}b^t$ and by $[B, t]$ the subgroup $B^{(t-1)}$ of B .

We may regard Y as a multiplicatively closed subset of T . If $y \in Y$ then for any positive integer r , $y^r - 1$ can be factorised as $P(y)(y-1)$ for some polynomial $P(y)$. Since y has finite order, it is easily seen that the subgroup $[B, \langle y \rangle]$ of G is just $[B, y]$.

For each positive integer m , choose an element y_m in Y of order p^m . Then if Y_m denotes $\langle y_m \rangle$, we have

$$\begin{aligned} \gamma_{BY_m}^{p^m} &= B^{(y_m - 1)^{p^m}} \\ &= B^{y_m^{p^m} - 1} = 1, \end{aligned}$$

but

$$\gamma_{BY_m}^{p^{m-1}} = B^{y_m^{p^{m-1}} - 1} \neq 1.$$

This means that, since $G = YB$ and Y is abelian,

$$\gamma_{GY_m}^{p^m} = 1,$$

but

$$\gamma_{GY_m}^{p^{m-1}} \cap B \neq 1.$$

It follows that each Y_m is subnormal in G , and that

$$p^{m-1} < s(G : Y_m) \leq p^m.$$

Thus there is certainly no bound for the defects of the subnormal subgroups of G .

Now we show that, for any subgroup H of G , the standard series of H in G becomes stationary after finitely many steps. If $HB < G$, then $HB = Y_m B$ for some positive integer m , or else $HB = B$. If $HB = Y_m B$, then HB is the join of a normal abelian subgroup B and a subnormal abelian subgroup Y_m , so it is nilpotent by Lemma 2.22. H is then subnormal in HB , which is normal in G , so the standard series of H in G reaches H after finitely many steps. If $HB = B$ the same is true. The only outstanding possibility is $HB = G$.

If this is the case, consider $[B, Y]$. This subgroup is generated by elements of the form $b^{(y-1)}$ with $b \in B$, $y \in Y$. Now $y = y_1^p$ for some $y_1 \in Y$, thus

$$b^{(y-1)} = b^{y_1^p-1} = b^{(y_1-1)^p} \in \gamma_{BY}^p.$$

It follows that $[B, Y] = \gamma_{BY}^p$, indeed that

$$[B, Y] = [B, Y, Y] .$$

Now

$$\begin{aligned} [B, H] &= [B, BH] \quad \text{since } B \text{ is abelian} \\ &= [B, BY] \quad \text{since } BH = G = BY \\ &= [B, Y] \quad \text{since } B \text{ is abelian.} \end{aligned}$$

Similarly $[B, H, H] = [B, Y, Y]$, so that $[B, H, H] = [B, H]$.

Hence

$$\begin{aligned} H^{G,1} &= H[G, H] \\ &= H[B, H] \\ &= H[B, H, H] = H^{G,2} . \end{aligned}$$

Thus even in the case $HB = G$, the standard series of H in G is well-behaved. Hence G has the subnormal intersection property, by Lemma 2.41.

A substantial part of this thesis is taken up with the question: for what groups does the possession of the subnormal intersection property imply the existence of a bound on the subnormal indices? Example 2.44 shows that this implication does not hold even for metabelian p -groups.

Lemma 2.43 ensures that a group satisfying the minimal condition on subnormal subgroups will have the subnormal intersection property. (For the record, a group satisfies the minimal (maximal) condition on subgroups of a given kind if every descending (ascending) chain of subgroups of that kind terminates after finitely many steps.) Indeed,

we can prove the following stronger result.

2.45 LEMMA. *If G has a normal subgroup N which satisfies the minimal condition on subnormal subgroups, and G/N has the subnormal intersection property, then G has this property also.*

Proof: Let H be any subgroup of G . By Lemma 2.41 and remark 1.23 (c), there is an integer $k \geq 0$ such that

$$H^{G,k}_N = H^{G,k+1}_N, \text{ since } G/N \text{ has the subnormal intersection}$$

property. But since N satisfies the minimal condition on subnormal subgroups, there is an integer $m \geq k$ such that

$$H^{G,m} \cap N = H^{G,m+1} \cap N. \text{ Since } H^{G,m}_N = H^{G,m+1}_N \text{ is also true, by a}$$

familiar argument we have $H^{G,m} = H^{G,m+1}$. This proves the result, by Lemma 2.41.

REMARK. In [25] (Lemma 3.2) Robinson proves a stronger result than that mentioned in the context of Lemma 2.45, namely that a group which satisfies the minimal condition on subnormal subgroups even has a bound for its subnormal indices. This somewhat surprising result will be further strengthened in 4.43, where we will have enough machinery available to prove the obvious analogue of Lemma 2.45.

2.5 Joins of Subnormal Subgroups

We conclude this chapter with a discussion of the theory of joins of subnormal subgroups, as propounded by Robinson in [23]. The treatment will be sketchy, as this topic occupies only a peripheral position in the thesis.

In contrast to the situation for intersections, it is not true

that in any group the join of two subnormal subgroups is a subnormal subgroup. In Theorems 6.1 and 6.2 of [23] Robinson constructs a soluble group of derived length 3 and a finitely-generated soluble group of derived length 4 in which this fails to happen.

However, the class of groups in which the join of two subnormal subgroups is always a subnormal subgroup is too large to be of much concern to us here. Indeed most of the groups with which we deal lie in the much narrower class of groups in which the join of an arbitrary family of subnormal subgroups is a subnormal subgroup. We will say that the groups in the latter class have the *subnormal join property*.

Example 2.44 provides us with a group which does not have the subnormal join property, for although (in the notation of 2.44) each of the subgroups Y_m is subnormal, their join $Y = U\{Y_m : m \geq 1\}$ cannot be a subnormal subgroup, for if it were G would be nilpotent by Lemma 2.22. This group, therefore, has the subnormal intersection property but not the subnormal join property. We will see from 2.51 below that any group satisfying the maximal condition on subgroups has the subnormal join property. Thus the infinite dihedral group (2.35) has this property but fails to have the subnormal intersection property.

If a group G has the property that the join of any two subnormal subgroup is a subnormal subgroup, we can consider the set of all subnormal subgroups of G as a lattice under the operations of intersection and join, that is, as a sub-lattice of the lattice of all subgroups of G . In the event that G has the subnormal intersection property and the subnormal join property, the set of all subnormal subgroups of G becomes a complete sub-lattice of the lattice of all

subgroups of G . Groups with these properties need not have a bound on their subnormal indices: for instance, the Heineken-Mohamed example ([14]) mentioned before 2.36 has both these properties, but by Theorem 2.36 it cannot have a bound for the defects of its subnormal subgroups, since it is not nilpotent. However, groups with these two properties do behave somewhat similarly to groups with bounded subnormal indices, as evidenced, for example, in the attached paper [A].

We state, without proof, a basic lemma.

2.51 LEMMA ([23], Lemma 8.1). *A group G has the subnormal join property if and only if the union of any ascending chain of subnormal subgroups of G is a subnormal subgroup of G .*

Since our next lemma, though well-known, is not proved in the literature, we include a proof.

2.52 LEMMA. *A group with a bound on its subnormal indices has the subnormal join property.*

Proof: Let G be a group with a bound r on its subnormal indices. Let $\{H_i : i \in I\}$ be an ascending chain of subnormal subgroups of G , and let H be their union. We prove by induction that for each non-negative integer n ,

$$\gamma_{GH}^n \leq \bigcup \left\{ \gamma_{GH_i}^n : i \in I \right\}.$$

This is trivially true for $n = 0$. If $n > 0$ and the statement is true for $n - 1$, we note that γ_{GH}^n is precisely

$$\langle [g, h] : g \in \gamma_{GH}^{n-1}, h \in H \rangle ,$$

so that it will suffice to show that each element of the form

$$[g, h] : g \in \gamma_{GH}^{n-1}, h \in H ,$$

lies in some $\gamma_{GH_i}^n$. But by the induction hypothesis, $g \in \gamma_{GH_j}^{n-1}$ for some $j \in I$; also $h \in H_k$ for some $k \in I$. There is an element $m \in I$ such that $\langle h, H_j \rangle \leq H_m$, for either $H_j \leq H_k$ or else $H_k \leq H_j$. It is then clear that $[g, h] \in \gamma_{GH_m}^n$ as required.

Now it follows that

$$\gamma_{GH}^r \leq \cup \left\{ \gamma_{GH_i}^r : i \in I \right\} \leq \cup \{ H_i : i \in I \} ,$$

since each H_i is subnormal of defect at most r . That is, $\gamma_{GH}^r \leq H$, so H is subnormal in G . By 2.51, G has the subnormal join property.

Finally we prove a lemma which will be of use in later chapters.

2.53 LEMMA ([23], Lemma 2.2). *Let H and K be subnormal subgroups of a group G . If $H^K = H$, then $J = \langle H, K \rangle = HK$ is subnormal in G and $s(G : J) \leq s(G : H)s(G : K)$.*

Proof: Let r and s be the defects of H and K respectively. Denote by H_i the subgroup $H^{G,i}$. Then if $H_i^K = H_i$, it follows that

$$(H_{i+1})^K = H_{i+1}^{H,K} = H_{i+1}^{KH} = H_{i+1}^{H,K} = H_{i+1}^H,$$

using $H^K = H$. Since $H_0 = G$ it is clear that each term H_i in the standard series of H in G is normalised by K . Thus for each i with $0 \leq i < r$, H_{i+1} is normal in H_i^K , and so by remark 2.21 (c), H_{i+1}^K has defect at most s in H_i^K . A simple summation then shows that $J = HK$ is subnormal in G of defect at most rs , as required.

3.1 Series and central series

A basic concept of infinite group theory is that of a series or normal system in a group. (See, for example, [1, p. 174].) In this section we present a treatment which follows that of Hall [1].

3.11 DEFINITION. Let G be a group and Ω a linearly ordered set. By a series in G of type Ω we mean a sequence

$$(*) \quad \Lambda_0, \Lambda_1, \Lambda_2, \dots$$

of pairs of subgroups of G , with the following properties:

- (i) Λ_α is a normal subgroup of Λ_β for each $\alpha < \beta$;
- (ii) $\Lambda_\alpha \neq \Lambda_\beta$ whenever $\alpha < \beta$;
- (iii) $\alpha < \beta \Rightarrow (\Lambda_\alpha \cap \Lambda_\beta) \neq \Lambda_\beta$.

Here, if X and Y are subsets of G , $X - Y$ denotes the usual set-theoretic difference. These conditions imply that a given

CHAPTER 3

 π -TORSION-FREENESS AND π -RADICABILITY

In this chapter we introduce and develop several useful concepts which enable us to prove a sequence of lemmas needed in later work. The first section is devoted to a general discussion of series in a group, with specific reference to ZA-groups. In Section 3.2 we discuss π -torsion-freeness in some restricted situations. Sections 3.3 and 3.4 deal with quasi- π -radicable and π -radicable groups, and their place in the theory of subnormal subgroups. Finally, in Section 3.5, we prove the promised succession of lemmas. These results, mainly concerned with automorphisms of groups with the properties discussed in 3.2 and 3.3, will be of fundamental importance in Chapters 5 and 6. In the early parts of the chapter many of the proofs are omitted for brevity.

3.1 Series and central series

A basic concept of infinite group theory is that of a series or normal system in a group. (See, for example, [17], p. 171). In this section we present a treatment which follows that of Hall [11].

3.11 DEFINITION. Let G be a group and Ω a linearly ordered set. By a *series in G of type Ω* we mean a set

$$(*) \quad \{\bigwedge_{\sigma}, \bigvee_{\sigma} : \sigma \in \Omega\}$$

of pairs of subgroups of G , with the following properties:

- (i) \bigvee_{σ} is a normal subgroup of \bigwedge_{σ} for each σ in Ω ;
- (ii) $\bigwedge_{\sigma} \leq \bigvee_{\tau}$ whenever $\sigma < \tau$;
- (iii) $G - 1 = \bigcup \{(\bigwedge_{\sigma} - \bigvee_{\sigma}) : \sigma \in \Omega\}$.

Here, if X and Y are subsets of G , $X - Y$ denotes the usual set-theoretic difference. These conditions imply that a given

non-identity element of G lies in exactly one of the "layers"
 $(\bigwedge_{\sigma} - \bigvee_{\sigma})$.

Another easy consequence of the conditions is

(iv) for each $\sigma \in \Omega$,

$$(a) \quad \bigvee_{\sigma} = \bigcup \{ \bigwedge_{\tau} : \tau < \sigma \} ;$$

$$(b) \quad \bigwedge_{\sigma} = \bigcap \{ \bigvee_{\tau} : \tau > \sigma \} .$$

The series (*) is said to be *invariant* when, for each σ in Ω ,
 \bigwedge_{σ} is a normal subgroup of G , or, equivalently, when, for each σ
in Ω , \bigvee_{σ} is a normal subgroup of G .

The series (*) is said to be *central* when, for each σ in Ω ,

$$[\bigwedge_{\sigma}, G] \leq \bigvee_{\sigma} .$$

Clearly any central series is invariant.

The more familiar concepts of ascending and descending series can
be obtained as special cases of this general type of series.

(A) If Ω is well-ordered then (*) becomes an ascending series.

In this case we may take Ω to be the set of all ordinals $\sigma < \rho$ for
the ordinal ρ which is the order-type of Ω . Then we have

$$\bigwedge_{\sigma} = \bigvee_{\sigma+1} \quad \text{for } \sigma + 1 < \rho , \text{ and if we define } \bigvee_{\rho} = G \text{ the } \bigwedge_{\sigma}$$

become superfluous, and the series takes the form

$$\{ \bigvee_{\sigma} : \sigma \leq \rho \} ,$$

with $\bigvee_0 = 1$, \bigvee_{σ} normal in $\bigvee_{\sigma+1}$ for $\sigma < \rho$, $\bigvee_{\rho} = G$, and,

for every limit ordinal $\mu \leq \rho$,

$$\bigvee_{\mu} = \bigcup \{ \bigvee_{\sigma} : \sigma < \mu \} .$$

(B) If Ω is inversely well-ordered, then (*) becomes a
descending series. Again we may take Ω to be the set of all
ordinals $\sigma < \rho$, for some ordinal ρ , but we must replace condition
(ii) of definition 3.11 by

(ii)' $\bigwedge_{\sigma} \leq \bigvee_{\tau}$ whenever $\tau < \sigma$.

Then we have $\bigvee_{\sigma} = \bigwedge_{\sigma+1}$ for $\sigma + 1 < \rho$, and if we define $\bigwedge_{\rho} = 1$ the \bigvee_{σ} become superfluous, and the series takes the form

$$\{\bigwedge_{\sigma} : \sigma \leq \rho\},$$

with $\bigwedge_0 = G$, $\bigwedge_{\sigma+1}$ normal in \bigwedge_{σ} for each $\sigma < \rho$, $\bigwedge_{\rho} = 1$ and, for every limit ordinal $\mu \leq \rho$,

$$\bigwedge_{\mu} = \bigcap \{\bigwedge_{\sigma} : \sigma < \mu\}.$$

The notion of series will be used strongly in Section 5.4, but for this chapter we require it in the following well-known definition. (See [17], p. 218.)

3.12 DEFINITION. A *ZA-group* is a group which has an ascending central series.

We make two remarks on this definition.

- (a) Any subgroup or quotient group of a ZA-group is a ZA-group.
- (b) The centre of a ZA-group is non-trivial.

In any group G we can define an ascending chain of subgroups of G , its *upper central series*, by taking

$$\zeta_0(G) = 1,$$

$\zeta_{\alpha}(G)/\zeta_{\alpha-1}(G)$ to be the centre of $G/\zeta_{\alpha-1}(G)$ for each positive, non-limit ordinal α , and

$$\zeta_{\mu}(G) = \bigcup \{\zeta_{\alpha}(G) : \alpha < \mu\}$$

for each limit ordinal μ . There will of course be a first ordinal β , depending on G , such that $\zeta_{\beta}(G) = \zeta_{\beta+1}(G)$. The upper central series of G is then

$$\{\zeta_{\alpha}(G) : \alpha \leq \beta\}.$$

From (a) and (b) we see that the group G is a ZA-group if and only if, for this ordinal β , $\zeta_\beta(G) = G$, that is, the upper central series of G is an ascending series in the strict sense of 3.11 (A). (G is of course nilpotent if and only if this ordinal β is finite.)

In later stages of the chapter it will be convenient to have available some simple results on ZA-groups. The first of these is well known; for the (easy) proof we refer the reader to [29], (1.51).

3.13 LEMMA. *Let N be a non-trivial normal subgroup of a ZA-group G . Then $N \cap \zeta_1(G)$ is non-trivial. (Here $\zeta_1(G)$ is, as above, the centre of G .)*

Our next lemma is a well-known result of Mal'cev. For the (not so easy) proof see [17], p. 223.

3.14 LEMMA ([18]). *A finitely-generated ZA-group is nilpotent.*

Before stating the last result of this section, it will be necessary to introduce some standard terminology which will be used throughout this and subsequent chapters.

We will denote by π a non-empty set of prime natural numbers, and by π' its complement in the set of all primes. A positive integer k will be called a π -number if each prime divisor of k lies in π : we may think of the set of π -numbers as the multiplicative sub-semigroup of the positive integers generated by the set $\pi \cup \{1\}$. In a group, an element will be termed a π -element if its order is a π -number. A group in which every element is a π -element will be called a π -group; if no element, other than 1, is a π -element the group will be said to be π -torsion-free. If π consists of only one prime, p say, we will use such terms as p -group, p -torsion-free. Indeed a group is π -torsion-free if and only if it is p -torsion-free for each p in π .

The following result is a simple consequence of 3.14 and the elementary properties of nilpotent groups (see, for example, Theorem 1.10 of [9]).

3.15 LEMMA. *Let G be a ZA-group and π a non-empty set of primes. Then the set of π -elements of G is a (fully invariant) subgroup of G .*

In any group, this subgroup, if it exists, is called the π -torsion subgroup.

3.2 π -torsion-freeness

In this short section we record, for easy reference, some basic results on π -torsion-free groups. The first of these is due to Mal'cev; a proof can be found in [29], (1.63).

3.21 LEMMA. *Let G be a group whose centre $\zeta_1(G)$ is π -torsion-free for some non-empty set of primes π . Then for any ordinal α , $\zeta_{\alpha+1}(G)/\zeta_\alpha(G)$ is π -torsion-free.*

A simple transfinite induction leads to the following corollary.

3.22 COROLLARY. *Let G be a group with upper central series $\{\zeta_\alpha(G) : \alpha \leq \beta\}$.*

If $\zeta_1(G)$ is π -torsion-free for some non-empty set of primes π , then for any ordinals λ, μ with $\mu \leq \lambda \leq \beta$, $\zeta_\lambda(G)/\zeta_\mu(G)$ is π -torsion-free. In particular, if G is a ZA-group, G itself is π -torsion-free.

Although we will be mainly concerned with nilpotent (often abelian) π -torsion-free groups, we remain in the more general content of ZA-groups to state the next important result, which is an easy consequence of 3.21. A proof can be found on p. 247 of [17].

3.23 LEMMA. Let G be a ZA-group and π a non-empty set of primes. Then G is π -torsion-free if and only if, for each p in π , $x^p = y^p$ with x, y in G implies $x = y$.

Note that the condition could be replaced by the requirement that for each π -number k , $x^k = y^k$ with x, y in G implies $x = y$.

3.3 Quasi- π -radicability

Before making the first definition of this section, we need to introduce some more notation. If G is a group and k a positive integer, we will denote by G^k the subgroup $\langle g^k : g \in G \rangle$ of G . For each k , G^k is a fully invariant subgroup of G .

We are now in a position to define and compare two important standard concepts.

3.31 DEFINITION. If π is a non-empty set of primes, a group G is said to be *quasi- π -radicable* if, for each π -number k , $G = G^k$, that is, each element of G can be expressed as a product of k -th powers.

3.32 DEFINITION. If π is a non-empty set of primes, a group G is said to be *π -radicable* if, for each π -number k , each element of G is the k -th power of some element of G .

We remark that when π is the set of all primes we will use the terms "quasi-radicable" ("Černikov complete") and "radicable" ("complete" or, for abelian groups, "divisible").

It is clear that Definition 3.32 loses none of its force if we replace "for each π -number k " by "for each prime p in π ". In contrast to this, Definition 3.31 cannot be weakened in this way without loss. For example, if F denotes the simple group of order 60, then F is quasi- p -radicable (where we omit the braces) for each

prime p in $\pi = \{2, 3, 5\}$, because for each $r > 0$, F^{p^r} is always a non-trivial normal subgroup of F . But the fact that $F^{30} = 1$ shows that F is not quasi- π -radicable. The following simple lemma shows that if we restrict our attention to soluble groups this difficulty disappears.

3.33 LEMMA. *A soluble group G is quasi- π -radicable, for some non-empty set of primes π , if and only if $G = G^p$ for each p in π .*

Proof: Only the sufficiency of the condition is in question. Suppose $G = G^p$ for each p in π , and let k be any π -number. If we write $H = G/G^k$ then H is a soluble group of exponent dividing k and $H = H^p$ for each p in π . Then H/H' is an abelian group with the same properties as H , so H/H' is trivial. Hence H is trivial and $G = G^k$, proving the lemma.

For soluble groups the criterion of 3.33 will be used (often implicitly) as a substitute for Definition 3.31.

We now state a trivial, but useful, characterisation of quasi- π -radicability. (Both 3.34 and 3.33 are presumably well-known.)

3.34 LEMMA. *A group G is quasi- π -radicable, for some non-empty set of primes π , if and only if G has no proper normal subgroup N such that the exponent of G/N is a π -number.*

It is clear from a comparison of Definitions 3.31 and 3.32 that π -radicability implies quasi- π -radicability. Indeed it is easy to see that a group G is π -radicable if and only if it is quasi- π -radicable and, for each π -number k , the set $\{g^k : g \in G\}$ is a subgroup of G . However, the two properties are not the same, as shown by the

following examples.

3.35 EXAMPLE ([29], (6.4)). Let G be the standard restricted wreath product of two quasi-cyclic p -groups, for a given prime p . Clearly G has no proper quotients of finite exponent and so, by 3.34, is quasi-radicable. However it is routine to show (see [29]) that G is not even p -radicable.

3.36 EXAMPLE. If p is an odd prime, define a group G_p by

$$G_p = \langle a, b : a^p = b^2 = (ab)^2 = 1 \rangle.$$

Since $b = b^p$ and $ab = (ab)^p$, G_p is quasi- p -radicable. But a can have no p -th root since $|G_p| = 2p$, so G_p is not p -radicable.

The proof of our next lemma involves a much-used technique.

3.37 LEMMA. Let π be a non-empty set of primes for which π' is also non-empty. Then if G is a π' -group, G is π -radicable; and conversely, if G is a π -radicable group of finite exponent k , k must be a π' -number.

Proof: Let m be a π -number. Then for any g in G , the order r of g is coprime to m , as G is a π' -group. Thus there are integers s and t such that $sr + tm = 1$. Then

$$g = g^{sr+tm} = (g^t)^m,$$

that is, g is an m -th power. This proves that G is π -radicable.

Conversely, if G is π -radicable and some prime p in π divides the exponent k of G , then $k = pn$ for some integer n with $0 < n < k$. Now for each g in G there is an element h of G with $g = h^p$. Then

$$g^n = h^{pn} = h^k = 1,$$

contradicting the fact that k is the exponent of G . k is thus a

π' -number.

The following important theorem, a generalisation of a result of Černikov, shows that for ZA-groups the situations arising in examples 3.35 and 3.36 cannot occur. We will omit the proof for brevity, since it is an easy generalisation of that given in [29], Theorem (6.41) (see also [17], pp. 234, 238).

3.38 THEOREM. *Let G be a quasi- π -radicable ZA-group, for some non-empty set of primes π . Then*

- (i) G is π -radicable;
- (ii) the π -torsion subgroup of G is π -radicable and lies in the centre of G ;
- (iii) if $\zeta_\alpha(G)$ and $\zeta_\beta(G)$ are any two terms of the upper central series of G , with $\alpha \geq \beta$, then $\zeta_\alpha(G)/\zeta_\beta(G)$ is π -radicable.

There is a very useful corollary.

3.39 COROLLARY. *A quasi- π -radicable ZA-group which is a π -group is abelian.*

3.4 π -radicability

In this section we continue our investigations of quasi- π -radicability and π -radicability.

In any group G , the join of any family of quasi- π -radicable subgroups is again a quasi- π -radicable subgroup, for obvious reasons. In contrast, a similar statement for π -radicable subgroups does not hold, for if p is an odd prime the group G_p of Example 3.36 is the join of two p -radicable subgroups $\langle b \rangle$ and $\langle ab \rangle$, yet G_p is not p -radicable. The group of Example 3.35 is the join of two radicable subgroups B and Y but is not radicable.

Our previous remarks indicate that, for each non-empty set of primes π , every group G has a unique maximal quasi- π -radicable subgroup $Q(\pi)$, which is clearly a fully invariant subgroup of G , since each of its quotients remains quasi- π -radicable. Moreover the trivial fact that an extension of a quasi- π -radicable group by another is a quasi- π -radicable group forces us to conclude that $G/Q(\pi)$ has no non-trivial quasi- π -radicable subgroups. A group with the latter property, namely that its maximal quasi- π -radicable subgroup is trivial, is termed π -reduced. If π is the set of all primes, we use the term reduced.

Although we have seen that in general a group need not have a unique maximal π -radicable subgroup, Theorem 3.38 ensures that in the case of a ZA-group such a subgroup will exist.

The first lemma of this section records, for reference, some simple connections between these ideas and those of 3.2.

3.41 LEMMA. *If π is a non-empty set of primes, and $Q(\pi)$ is the maximal quasi- π -radicable subgroup of a group G , then $G/Q(\pi)$ is π' -torsion-free. If, in addition, G is a π -torsion-free ZA-group, $G/Q(\pi)$ is in fact torsion-free.*

Proof: Since any π' -group is π -radicable by 3.37, and $G/Q(\pi)$ is π -reduced, the first part is immediate. To prove the second, we need only show that $G/Q(\pi)$ is π -torsion-free. Suppose, therefore, that $x \in G$ with $x^k \in Q(\pi)$ for some π -number k . Now $Q(\pi)$ is π -radicable by Theorem 3.38, so there is an element y in $Q(\pi)$ such that $x^k = y^k$. By Lemma 3.23 we have $x = y$, that is $x \in Q(\pi)$. This shows that $G/Q(\pi)$ is π -torsion-free, as required.

It is convenient in this context to include a useful result due to D. McDougall. It appears as Lemma 3.6 of the attached paper [A],

to which we refer the reader for a proof.

3.42 LEMMA. *Let N be a nilpotent group with upper central series $\{\zeta_i(N) : 0 \leq i \leq c\}$. Let π be a non-empty set of primes. Define R_i inductively, for each $i \geq 0$, by: $R_0 = 1$, R_i/R_{i-1} is the maximal π -radicable subgroup of the centre of N/R_{i-1} . Then R_i is the maximal π -radicable subgroup of $\zeta_i(N)$, and in particular $R_c = R_{c+1}$.*

It is perhaps surprising that this result does not hold for ZA-groups, as shown by our next example, which will be useful in another context.

3.43 EXAMPLE. A quasi-cyclic 2-group A is the union of an ascending chain of cyclic subgroups $\{A_n : n \geq 0\}$ where $|A_n| = 2^n$. A has an automorphism x of order 2, inverting each element. If G denotes the split extension of A by $\langle x \rangle$ with this action, it is seen that in the upper central series of G , $\zeta_n(G) = A_n$ for each $n \geq 0$, $\zeta_\omega(G) = A$ where ω is the first infinite ordinal, and $\zeta_{\omega+1}(G) = G$. Thus G is a ZA-group. But if we take $\pi = \{2\}$ and attempt to define a chain $\{R_\alpha : \alpha \leq \omega\}$ by defining R_n as in 3.42 for each integer $n \geq 0$, and $R_\omega = \bigcup \{R_n : n < \omega\}$, we see that R_ω is trivial although $\zeta_\omega(G) = A$ is 2-radicable.

We now give a useful result which does hold good for ZA-groups. Its proof will be omitted, since it runs almost exactly as in Lemma 3.4 of the attached paper [A].

3.44 LEMMA. *If G is a π -reduced ZA-group for some non-empty set of primes π , then $G/\zeta_1(G)$ is π -reduced.*

To indicate how π -radicable subgroups fit into the structure of abelian groups, we include an easy generalisation of a well-known result.

3.45 THEOREM ([6], pp. 62-63). *Let A be an abelian group and π a non-empty set of primes. If P is any π -radicable subgroup of A , then there is a subgroup B of A such that $P \cap B = 1$ and A/BP is a π' -group. Moreover B can be chosen so as to contain an arbitrary subgroup C of A with $C \cap P = 1$.*

Perhaps a more useful result in the context of subnormality is the following lemma of Robinson. We will omit the rather lengthy proof since it runs essentially as in [25], Lemma 2.1.

3.46 LEMMA (cf [26], Lemma 4). *Let A be a π -radicable abelian normal subgroup of a group G , for some non-empty set of primes π . Let H be a subnormal subgroup of G such that H/H' is a π -group. Then H is normal in HA . If, in addition, H is nilpotent then $[H, A] = 1$.*

Finally we point out that the proof of Lemma 3.7 of [A] now carries over almost verbatim to give:

3.47 LEMMA. *Let π be a non-empty set of primes. Let the group G have a π -radicable normal subgroup N which is nilpotent of class c . If H is a subnormal subgroup of G such that H/H' is a π -group then $s(HN : H) \leq c$.*

3.5 Five easy lemmas

In this section we gather together some useful (possibly well-known) results involving the ideas introduced in earlier sections of this chapter. We need an additional piece of terminology: if H acts as a group of automorphisms of a group G , we will say that H acts *fixed-point-freely* on G if the centraliser in G of H is

trivial, that is, if

$$C_G(H) = \langle g \in G : g^h = g \text{ for each } h \in H \rangle = 1.$$

3.51 LEMMA. *Let π be a non-empty set of primes. If A is a π -torsion-free abelian group acted on by a group of automorphisms H which is a π -group, and $C = C_A(H)$, then A/C is a π -torsion-free group on which H acts fixed-point-freely.*

Proof: If A/C is not π -torsion-free, then there is an element $a \in A - C$ such that $a^p \in C$ for some prime p in π . Thus for any $h \in H$,

$$[a, h]^p = [a^p, h] = 1,$$

showing that $[a, h] = 1$ since A is π -torsion-free. This holds for each $h \in H$, so $a \in C$, a contradiction.

It is clear that H can be regarded as a group of automorphisms of A/C . Suppose $bC \in A/C$ is fixed under the action of H . Then for any $h \in H$, $b^h = bc$ for some $c \in C$. By repeatedly applying the automorphism h , we see that, for any positive integer r ,

$$b^{h^r} = bc^r.$$

Now the order of h is some π -number k , hence

$$b = b^{h^k} = bc^k,$$

and $c^k = 1$. A is π -torsion-free, so $c = 1$ and $b^h = b$. This holds for any $h \in H$, so $b \in C$, proving that the action of H on A/C is indeed fixed-point-free.

Now we give another very simple result in the same vein.

3.52 LEMMA. *If H is a group of order k which acts fixed-point-freely as a group of automorphisms of an abelian group A , then*

$$A^k \leq [A, H] = \langle [a, h] : a \in A \ h \in H \rangle.$$

Proof: Let $H = \{h_1, \dots, h_k\}$. Then for any a in A , the element

$$a_1 = a^{h_1} \dots a^{h_k}$$

is fixed by each $h_i \in H$, since A is abelian. By assumption, then, $a_1 = 1$. But

$$a_1 = a^k [a, h_1] \dots [a, h_k].$$

This shows that $a^k \in [A, H]$ for each $a \in A$, that is, $A^k \leq [A, H]$.

The following useful result is an immediate corollary.

3.53 COROLLARY. *Let π be a non-empty set of primes. If H is a group of order a π -number which acts fixed-point-freely as a group of automorphisms of a π -radicable abelian group A , then $[A, H] = A$.*

In some circumstances it is necessary to know whether fixed-point-free action is "transferred" to factor groups. Our next result gives some conditions under which this occurs.

3.54 LEMMA. *Let π be a non-empty set of primes. If H is a group of order a π -number k , which acts fixed-point-freely as a group of automorphisms of an abelian group A , and if B is an H -invariant subgroup of A such that A/B is π -torsion-free, then the action of H on A/B remains fixed-point-free.*

Proof. Let $H = \{h_1, \dots, h_k\}$. If the element aB of A/B is fixed under the action of H , we have, for each $h_i \in H$,

$$a^{h_i} B = aB$$

where $b_i \in B$. Then

$$a_1 = a^{h_1} \dots a^{h_k} = a^k_b$$

for some $b \in B$. But, as in 3.52, a_1 is fixed under the action of H , so $a_1 = 1$ and $a^k \in B$ follows. Since A/B is π -torsion-free, $a \in B$ and the lemma is proved.

We now combine two previous results to obtain a lemma which will be invaluable in Chapters 5 and 6.

3.55 LEMMA. *Let π be a non-empty set of primes. If a group H of order a π -number acts as a group of automorphisms on a π -radicable, π -torsion-free abelian group A , then A is the product of two subgroups C and D with $[C, H] = 1$ and $[D, H] = D$. Moreover, if H is cyclic, then $C \cap D = 1$.*

Proof: Choose $C = C_A(H)$. Then, by Lemma 3.51, A/C is acted on fixed-point-freely by H , and so by Corollary 3.53, $[A/C, H] = A/C$ or $[A, H]C = A$. If we take $D = [A, H]$ then $A = CD$ and

$$[D, H] = [A, H, H] = [[A, H]C, H] = [A, H] = D.$$

If $H = \langle h \rangle$, any element $b \in C \cap D$ has the form $[a, h]$ for some $a \in A$. (cf. 2.44.) Since $[a, h] \in C$, the element aC of A/C is fixed under the action of H , hence $a \in C$. Then $b = [a, h] = 1$ and we have shown $C \cap D = 1$.

(As Roger Bryant has pointed out, a more sophisticated proof will show that $C \cap D = 1$ even when H is not cyclic. However, for the purposes of this thesis, this stronger statement will not be needed.)

Now we state, without proof, a lemma and a corollary which are proved, with trivial changes, as 3.1 and 3.2 of the attached paper [A].

3.56 LEMMA. *Let x be an automorphism of order p^r of an*

abelian group A , where p is a prime and r a non-negative integer. Define an endomorphism ϕ of A by $a\phi = [a, x]$ for each $a \in A$. Then

$$A\phi^{p^r} \leq A^p.$$

3.57 COROLLARY. If, in the situation of Lemma 3.56, there is a subgroup B of A with $B\phi = B$, then B is p -radicable.

The reason for the inclusion of these two results is to enable us to prove, as a further corollary, a more detailed version of Lemma 3.3 of [A].

3.58 COROLLARY. Let x be an automorphism of order p^r of a p -reduced abelian group A , where p is a prime and r a non-negative integer. Let $A_0 = A$. Define $A_\alpha = [A_{\alpha-1}, x]$ for each positive, non-limit ordinal α , and

$$A_\lambda = \cap \{A_\alpha : \alpha < \lambda\}$$

for each limit ordinal λ . Then

- (i) $A_\beta = 1$ for some ordinal β ;
- (ii) if $A_k = A_{k+1}$ for some integer $k \geq 0$, then $A_k = 1$;
- (iii) if A is p -torsion-free in case (ii), then $A_1 = 1$.

Proof: (i) There is an ordinal β such that $A_\beta = A_{\beta+1}$.

Then, in the notation of 3.56, $A_\beta\phi = A_\beta$, so that by Corollary 3.57, A_β is p -radicable. But A is p -reduced, so $A_\beta = 1$.

(ii) is immediate from (i).

(iii) Suppose $A_1 \neq 1$ and let k be the least integer in the set $\{i : A_i = 1\}$. Then $k > 1$. Write $C = C_A(x)$. Clearly $A_{k-1} \leq C$, that is, $[A_{k-2}, x] \leq C$. This shows that the subgroup

$A_{k-2}^{C/C}$ of A/C is fixed under the action of $\langle x \rangle$. By Lemma 3.51, since A is p -torsion-free, $A_{k-2}^{C/C}$ must be trivial and $A_{k-2} \leq C$ follows. Then $A_{k-1} = [A_{k-2}, x] = 1$, contradicting the choice of k . Hence $A_1 = 1$ and the corollary is proved.

CHAPTER 4

SOME FINITENESS CONDITIONS FOR SOLUBLE GROUPS

In this chapter we introduce and discuss various finiteness conditions on soluble groups which will be relevant in the context of Chapter 6. In Section 4.1, after some preliminary definitions and lemmas, we define the torsion-free rank and (for each prime p) the p -rank of an abelian group A , noting that these, and related ranks, are invariants of A . From there we proceed to discuss in Section 4.2 some classes of abelian groups defined by restrictions on rank: we also introduce the class of abelian minimax groups. The next section extends the restrictions treated in 4.2 to soluble groups of arbitrary derived length. Finally, Section 4.4 contains some technical lemmas which will be required in Chapter 6, together with an interesting result promised in Chapter 2.

4.1 Rank

In this section we follow fairly closely the treatment of [6] pp. 29ff., leading up to the definitions and elementary properties of various ranks for abelian groups. Here, and throughout the next section, abelian groups will be written additively.

4.11 DEFINITION ([6], p. 29). Let A be an abelian group. A finite set $\{a_1, \dots, a_k\}$ of non-zero elements of A is said to be *independent* if, for any integers n_1, \dots, n_k , the equation

$$n_1 a_1 + \dots + n_k a_k = 0$$

implies

$$n_1 a_1 = \dots = n_k a_k = 0.$$

An arbitrary subset of A is said to be independent if each of its finite subsets is independent.

A subset of A which does not have this property is said to be

dependent.

If $a \in A$ and A_1 is an arbitrary subset of A , a is said to depend on A_1 if there is a finite subset $\{a_1, \dots, a_k\}$ of A_1 and integers n, n_1, \dots, n_k such that

$$0 \neq na = n_1 a_1 + \dots + n_k a_k.$$

There follows a trivial consequence of the definition.

4.12 LEMMA ([6], Lemma 8.1). A subset $\{a_\lambda : \lambda \in \Lambda\}$ of an abelian group A is independent if and only if the subgroup $\langle a_\lambda : \lambda \in \Lambda \rangle$ is the direct sum of the subgroups $\langle a_\lambda \rangle : \lambda \in \Lambda$ and no a_λ is 0.

It is then clear that an element a of an abelian group A depends on a subset $\{a_\lambda : \lambda \in \Lambda\}$ of A if and only if the subgroup $\langle a, a_\lambda : \lambda \in \Lambda \rangle$ is not the direct sum of $\langle a \rangle$ and $\langle a_\lambda : \lambda \in \Lambda \rangle$.

Now we deviate slightly from [6].

4.13 DEFINITION. Two subsets B and C of an abelian group A are said to be *mutually dependent* if each element of B depends on C and each element of C depends on B .

If the group A is torsion-free, the relation of mutual dependence is an equivalence relation on the set of subsets of $A - \{0\}$. Symmetry and reflexivity are obvious; transitivity follows by an argument which leans heavily on the torsion-freeness.

If A is not torsion-free, the relation may not be an equivalence. Let A be a cycle of order pq , where p and q are distinct primes. Then A has elements a of order p and b of order q . Now $\{a\}$ and $\{a+b\}$ are mutually dependent, as are $\{a+b\}$ and $\{b\}$. But $\langle a \rangle \cap \langle b \rangle = 0$, so that $\{a\}$ and $\{b\}$ are not mutually dependent.

The importance of the concept of mutual dependence becomes clear in the next lemma, which is proved on p. 30 of [6].

4.14 LEMMA. *Let B be a set of non-zero elements of an abelian group A . Then there is a subset B_1 of A in which each element of finite order is a p -element for some prime p , with B and B_1 mutually dependent. If B is independent then B_1 can be chosen to be independent also.*

REMARK. It is worthwhile pointing out that in 4.14 the subset B_1 can be chosen to have a stronger property than mutual dependence, in that each element of B lies in $\langle B_1 \rangle$, and conversely.

We now quote the useful Steinitz Exchange Lemma.

4.15 LEMMA ([6], Lemma 8.3). *Let A be a torsion-free abelian group with an independent subset $\{a_1, \dots, a_k\}$ each element of which depends on another subset $\{b_1, \dots, b_n\}$. Then $k \leq n$, and, for a suitable choice of subscripts, the set $\{a_1, \dots, a_k, b_{k+1}, \dots, b_n\}$ is equivalent to the set $\{b_1, \dots, b_n\}$.*

We are now in a position to define several types of rank for abelian groups.

4.16 DEFINITION. Let A be an abelian group. Then

- (a) the *total rank* $r(A)$ of A is the cardinal of a maximal independent set of elements (of A) of either infinite or prime-power order;
- (b) the *0-rank*, or *torsion-free rank*, $r_0(A)$ of A is the cardinal of a maximal independent set of elements (of A) of infinite order;
- (c) for each prime p , the *p -rank* $r_p(A)$ of A is the cardinal

of a maximal independent set of p -elements of A ;

- (d) the *reduced*, or *special rank* $r^*(A)$ of A is the cardinal $r_0(A) + \max\{r_p(A) : p \text{ any prime}\}$.

REMARKS. (i) The existence of subsets of A maximal with respect to the properties mentioned in (a), (b) and (c) follows from the fact that these properties are of finite character.

(ii) Any two maximal subsets occurring in (a), (b) or (c) will be mutually dependent.

(iii) If the subset B of Lemma 4.14 is a maximal independent set, then, by the remark following the lemma, the corresponding subset B_1 will be a maximal subset of the type mentioned in (a).

Before proceeding with our discussion of rank, we must convince ourselves that the cardinals mentioned in 4.16 are well-defined. This is not the case for maximal independent subsets: in the example of 4.13, $\{a+b\}$ and $\{a, b\}$ are both maximal independent subsets.

The proof that $r_0(A)$ is well-defined can be found in [6], pp. 31-33. If $r_0(A)$ is infinite, a cardinality argument suffices; if finite, Lemma 4.15 is needed. An alternative approach is to note that a maximal independent subset of elements of infinite order gives rise to a basis for the vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$, where \mathbb{Q} is the rational field and \mathbb{Z} the ring of rational integers.

To see that $r_p(A)$ is well-defined for each prime p , we first argue as in [6], p. 33, to show that $r_p(A) = r_p(A[p])$ where $A[p]$ is the subgroup $\{a \in A : pa = 0\}$. Then we may either follow the reasoning of [6] or note that $A[p]$ is a vector space over the Galois field $\text{GF}(p)$ and that a maximal independent subset of $A[p]$ is a

basis.

It is now clear that the reduced rank is well-defined.

Finally, we may apply Lemma 4.14 to deduce the equality

$$r(A) = r_0(A) + \sum \{r_p(A) : p \text{ any prime}\}$$

establishing that the total rank is well-defined.

To conclude this section we record some equalities and inequalities between various ranks, omitting the proofs (see [29], (6.11)).

4.17 LEMMA. *Let B be a subgroup of the abelian group A .*

Then

- (a) $r_0(A) = r_0(B) + r_0(A/B)$;
- (b) $r_p(A) - r_p(B) \leq r_p(A/B) \leq r_p(A) + r_0(A)$ for each prime p ;
- (c) $r(B) \leq r(A) \leq r(B) + r(A/B)$;
- (d) $r^*(B) \leq r^*(A) \leq r^*(B) + r^*(A/B)$.

4.2 Finiteness conditions for abelian groups

In this section we list and discuss some related finiteness conditions which may be satisfied by an abelian group A with torsion subgroup T .

4.21 (1): *A has finite 0-rank.*

It is easy to show that this occurs if and only if A/T is isomorphic to a subgroup of a direct sum of finitely many copies of the additive group of rational numbers. This property coincides with A_1 of [19] .

(2)(π) (where π , as usual, denotes a non-empty set of primes):
 A has finite p -rank for each prime p in π .

It is easy to prove that A has finite p -rank if and only if the p -torsion subgroup of A satisfies the minimal condition on subgroups.

(3): *A has finite 0-rank and finite p -rank for each prime p .*

This is generally abbreviated to " A has finite rank". If A has this property, T has property (2) with π the set of all primes, and A/T has property (1). The property A_2 of [19] demands that A/T have property (1) and that T be a direct product of cyclic and locally cyclic groups. Clearly A_2 is stronger than (1) but weaker than (3).

(4): A has finite reduced rank.

It can be shown that (4) is equivalent to the condition that, for some positive integer r , each finitely generated subgroup of A can be generated by r elements. Clearly (4) is a stronger property than (3).

(5): A has finite total rank.

From previous remarks it is clear that A has property (5) if and only if A/T has property (1) and T satisfies the minimal condition for subgroups. By 3.45 and 4.23 below, T is then the direct product of a finite number of cyclic and quasicyclic subgroups, which is condition A_3 of [19]. (5) is a stronger property than (4). Stronger still is Mal'cev's A_4 ([19]), which demands, in addition to (1), that T be finite.

It is not difficult to see from Lemma 4.17 that if A is an abelian group with one of the properties (1)-(5), A_2 or A_4 , then any subgroup of A has this property also. Moreover, if A has a subgroup B such that B and A/B have one of these properties, then A has this property also.

If A has one of the properties (1), (3), (4) or A_2 , then any factor group of A has this property. A free abelian group of infinite rank has property (2) for any set of primes π , but clearly has factor groups which do not. The additive group of rational

numbers has property A_4 (hence (5)) but its factor group with respect to any cyclic subgroup has an element of order p for each prime p , so it cannot have property (5).

The conditions so far described (with the exception of A_4) are all weaker than either the maximal or the minimal condition on subgroups. (The former is Mal'cev's A_5 ([19]).) Before stating our next condition, which is in a sense the most restrictive which is weaker than either of these two, it will be convenient to quote a well-known result of Černikov.

4.22 THEOREM ([4]). *A soluble group satisfying the minimal condition on subgroups is an extension of a periodic, radicable abelian group by a finite group.*

By 3.45 we have an immediate corollary.

4.23 COROLLARY. *An abelian group satisfying the minimal condition on subgroups is the direct sum of a characteristic radicable subgroup and a finite group.*

Our final condition is

(6): *A has a finite series*

$$0 = A_0 < A_1 < \dots < A_n = A$$

in which each factor A_i/A_{i-1} ($1 \leq i \leq n$) satisfies either the maximal or the minimal condition for subgroups.

We will say that A is an abelian *minimax* group, and the series above is called a *minimax* series.

If A has property (6) then any subgroup or factor group of A has property (6). If A has a subgroup B , such that A/B and B both have property (6), then A has property (6). Our previous remarks on the group of rational numbers show that this is not a

minimax group, so (6) is a strictly stronger property than (5).

We can simplify the structure of abelian minimax groups by the following lemma, which is easily proved using 4.23 and induction on the length of a minimax series.

4.24 LEMMA. *An abelian minimax group A has a subgroup M which satisfies the maximal condition, such that A/M satisfies the minimal condition.*

4.3 Finiteness conditions for soluble groups

In this section we extend the ideas of 4.2 to soluble groups in an obvious way.

4.31 DEFINITION. We will say that a group G is *soluble of type (k)* , where $1 \leq k \leq 6$, if G has a finite series

$$1 = G_0 < G_1 < \dots < G_n = G$$

of subgroups such that for $1 \leq i \leq n$ the factor G_i/G_{i-1} is an abelian group of type (k) as defined in Section 4.2. (This is for conciseness: we will generally use such phrases as " G is a soluble group of finite rank".)

The solubility of G in this definition is clear; moreover an abelian group which is soluble of type (k) will be of type (k) as in 4.2.

The following lemmas are easy consequences of remarks in 4.2.

4.32 LEMMA. *If G_1 and G_2 are soluble of type (k) ($1 \leq k \leq 6$) then any subgroup of G_1 , and any extension of G_1 by G_2 are soluble of type (k) .*

4.33 LEMMA. *If G is soluble of type (k) ($k = 1, 3, 4$ or 6) and N is a normal subgroup of G , then G/N is soluble of type (k) .*

4.34 LEMMA. *If G is soluble of type (5) (or (6)) and periodic, then G satisfies the minimal condition for subgroups.*

REMARK. Any periodic soluble group is soluble of type (1).

Our main concern in Chapter 6 will be with soluble groups of type (6), that is, soluble minimax groups. Further details on the structure of these groups may be found in [1] and [27].

4.4 Some results

In this section we provide two useful lemmas and prove a result which links this chapter to Chapter 2.

4.41 LEMMA. *Let G be a soluble group of finite torsion-free rank. If $\{H_i : i \geq 0\}$ is an ascending chain of normal subgroups of G , then all but finitely many of the factors $\{H_{i+1}/H_i : i \geq 0\}$ are periodic.*

Proof: We proceed by induction on the derived length of G . The statement is true when G is abelian, by a simple application of Lemma 4.17. Suppose it is true for groups of derived length less than k , where k is an integer greater than 1. Let G be a soluble group of finite torsion-free rank, with derived length k . Then by Lemmas 4.32 and 4.33 respectively, G' and G/G' are soluble groups of finite torsion-free rank and derived length less than k . If $\{H_i : i \geq 0\}$ is an ascending chain of normal subgroups of G , then $\{H_i \cap G' : i \geq 0\}$ and $\{H_i G' / G' : i \geq 0\}$ are similar chains in G' , G/G' respectively. By the induction hypothesis there is a non-negative integer m such that for $i \geq m$, both $(H_{i+1} \cap G') / (H_i \cap G')$ and $(H_{i+1} G' / G') / (H_i G' / G')$ are periodic. Then for $i \geq m$, $H_i (H_{i+1} \cap G') / H_i$, which is isomorphic to $(H_{i+1} \cap G') / (H_i \cap G')$, is

periodic. Also $H_{i+1}/H_i(H_{i+1} \cap G') = H_{i+1}/(H_{i+1} \cap (H_i G'))$ is isomorphic to $H_{i+1}G'/H_iG'$, hence is periodic. Thus H_{i+1}/H_i , which is an extension of $H_i(H_{i+1} \cap G')/H_i$ by $H_{i+1}/H_i(H_{i+1} \cap G')$, is periodic. This completes the proof of the lemma.

Our next lemma shows the relevance of finite rank to the theory of subnormal subgroups. The proof (which relies on 3.45) will be omitted.

4.42 LEMMA ([25], Lemma 2.1 (i)). *Let A be a radicable abelian normal subgroup and H a subnormal subgroup of a group G . If A has finite reduced rank r , then $s(HA : H) \leq r$.*

As an illustration of the use of Lemma 4.42, we prove the promised analogue of Lemma 2.45.

4.43 THEOREM. *Let G be a group with a normal subgroup N which satisfies the minimal condition on subnormal subgroups. If G/N has a bound for its subnormal indices, then so also has G .*

Proof: Suppose that G is a group which satisfies the conditions but not the conclusion of the theorem. Then in the family of normal subgroups (of G) $\{M_i : i \in I\}$ such that for each $i \in I$, $M_i \leq N$ and G/M_i has a bound for its subnormal indices, there must be minimal elements. Let M be one of these; then $M \neq 1$.

M clearly satisfies the minimal condition on subnormal subgroups. In particular it has a unique minimal subnormal subgroup M^* of finite index in M . M^* is characteristic in M , and G/M^* will have a bound for its subnormal indices by Lemma 2.34. If $M^* < M$, this would contradict the minimality of M . Thus M has no proper subgroups of finite index. But by Lemma 3.2 of [25], a group satisfying the minimal condition for subnormal subgroups has a subgroup

of finite index in which normality is transitive. Hence each subnormal subgroup of M is normal in M .

Secondly, if $M > M'$, M/M' is an abelian group satisfying the minimal condition on subgroups. Since M has no proper subgroups of finite index, it is readily seen (by 4.23) that M/M' is a radicable abelian group of finite reduced rank. By 4.42 this would imply that G/M' has a bound for its subnormal indices, in contradiction to the minimality of M . Thus $M = M'$, that is, M is perfect.

But Folgerung 1.2 of [15] states that in a group any perfect normal subgroup in which normality is transitive normalises every subnormal subgroup of the group. This implies that G has a bound for its subnormal indices, a contradiction which completes the proof.

5.1 Abelian-by-cyclic groups

Our investigations of normality in soluble groups begin in the very basic context of abelian-by-cyclic groups. First we prove a theorem based on a simple result of Chapter 3.

5.11 THEOREM. Let $G = A \rtimes C$, where A is an abelian normal subgroup of G , with $[A : C] = 1$ for some prime p and non-negative

CHAPTER 5

SOME RESULTS ON SUBNORMAL STRUCTURE

In this chapter we investigate the consequences, in some restricted situations, of supposing that a group has the subnormal intersection property. We show that in many cases this is equivalent to demanding that there should be a bound on the subnormal indices, a condition which, for the groups in question, is closely related to nilpotency in a sense which will become clear in later stages of the chapter.

In Section 5.1 we deal mainly with extensions of abelian groups by cyclic groups, showing that in some circumstances the restriction on intersections of subnormal subgroups enables us to determine the structure of the group in some detail. This early work relies heavily on the results of Chapter 3. We extend our consideration to abelian-by-nilpotent groups in Section 5.2 and to metanilpotent groups in Section 5.3. Throughout these sections we include simple examples to show why we restrict our investigations to the given situations, and to indicate what may go wrong if we slacken our requirements. Finally in Section 5.4 we apply the material of the preceding sections, together with a theorem of Hall and Hartley, to prove the interesting result that an abelian-by-finite group with the subnormal intersection property must have a bound on its subnormal indices.

5.1 Abelian-by-cyclic groups

Our investigations of subnormality in soluble groups begin in the very basic context of abelian-by-cyclic groups. First we prove a theorem based on a simple result of Chapter 3.

5.11 THEOREM. *Let $G = \langle x, A \rangle$, where A is an abelian normal subgroup of G , with $\left[x^{p^r}, A \right] = 1$ for some prime p and non-negative*

integer r . If G has the subnormal intersection property and A is p -reduced then G is nilpotent; if, in addition, A is p -torsion-free then G is abelian.

Proof: If M denotes the subgroup $\langle x \rangle \cap A$ of G , M is a central subgroup of G . In the group G/M the standard series of the subgroup $\langle xM \rangle$ is just

$$\{\langle x \rangle A_i / M : i \geq 0\},$$

where the A_i are defined recursively by: $A_0 = A$ and for $i \geq 0$,

$$A_{i+1} = [A_i, x] = [A_i, \langle x \rangle].$$

Now G/M inherits the subnormal intersection property, by Proposition 2.42, so that by Lemma 2.41 this standard series must become stationary after finitely many steps. This means that there is a positive integer k such that $\langle x \rangle A_{k-1} = \langle x \rangle A_k$. Then

$$A_{k+1} = [A_k, x] = [\langle x \rangle A_k, x] = [\langle x \rangle A_{k-1}, x] = [A_{k-1}, x] = A_k.$$

Now we can apply Corollary 3.58, since x acts as an automorphism of A of order dividing p^r . We deduce that $A_k = 1$.

But it is not difficult to see that $A_i = \gamma_{i+1}^G$ for each positive integer i , so that G is nilpotent of class at most k .

If A is p -torsion-free (indeed, since A is p -reduced, A will then be torsion-free) we deduce from (iii) of Corollary 3.58 that $A_1 = 1$ and hence that G is abelian.

We can immediately deduce a more general result.

5.12 THEOREM. Let π be a non-empty set of primes and let $G = \langle x, A \rangle$ where A is an abelian normal subgroup of G , with $[x^k, A] = 1$ for some π -number k . If G has the subnormal intersection property and A is p -reduced for each prime p in π ,

then G is nilpotent; if, in addition, A is π -torsion-free then G is abelian.

Proof: If we write $N = \langle x^k \rangle$ then N is central in G and G/AN has order k . G can then be written as the product of a finite number of normal subgroups G_q , where q ranges over the prime divisors of k , and G_q/AN is the Sylow q -subgroup of G/AN . Since k is a π -number, it is easy to see that each G_q satisfies the requirements of Theorem 5.11 and so is nilpotent. An application of Fitting's Theorem yields the desired conclusion.

If A is π -torsion-free (clearly a superfluous condition if π contains more than one prime) then by 5.11 each G_q is abelian and A is central in G . Since G/A is cyclic G must then be abelian.

We now discuss some examples which will show why we need the restrictions in Theorem 5.11, and will motivate further treatment of this situation.

5.13 EXAMPLE. Let A denote the additive group of all rational numbers of the form $m/2^n$, where m and n are integers. A is clearly 2-radicable, and has an automorphism x of order 2 defined by $a^x = -a$ for each a in A .

We consider the natural semidirect product of A and the 2-cycle $\langle x \rangle$ with this action. First we note that for any a in A ,

$$\left[-\frac{a}{2}, x \right] = \frac{a}{2} + \left(-\frac{a}{2} \right)^x = \frac{a}{2} + \frac{a}{2} = a,$$

showing that $[A, x] = A$. Now if S is any subnormal subgroup of G , either $SA = A$ or $SA = G$. In the first case S , being a subgroup of A , is normalised by x and so is normal in G . In the second case we have

$$[A, S] = [A, SA] = [A, G] = A ,$$

and we deduce that for each non-negative integer i ,

$$\gamma AS^i = A .$$

The subnormality of S then implies that S contains A , so that $S = G$. Thus any subnormal subgroup of G is actually normal in G , and G clearly has the subnormal intersection property.

Hence G satisfies all the conditions of Theorem 5.11, with the exception of the 2-reducedness of A , and G is certainly not nilpotent. Additional complexities may thus occur when, in Theorem 5.11, we allow A to have non-trivial p -radicable subgroups. However, although these complexities remove all prospect of the nilpotency of the group G , we will see later (Theorem 5.26) that they do not affect the existence of a bound on the subnormal indices in G .

5.14 EXAMPLE. For a given positive integer k , define a group G_k by

$$G_k = \langle a, b : a^{2^k} = b^2 = (ab)^2 = 1 \rangle .$$

Then $[a, b] = a^{-2}$, and, writing A for the subgroup $\langle a \rangle$, $[A, b] = A^2$. Indeed it is not difficult to see that for each positive integer r

$$\left[A, \underbrace{b, \dots, b}_r \right] = A^{2^r} .$$

It follows that $\gamma AG_k^k = 1$, whereas $\gamma AG_k^{k-1} > 1$, and G_k is nilpotent of class precisely k . But G_k satisfies all the conditions of Theorem 5.11 with the exception of the p -torsion-freeness of A (where $p = 2$). If we drop this latter requirement, therefore, the nilpotent class of the group G may be arbitrarily large.

In the sequel we often find it necessary to bound nilpotent classes in a given situation. We can do this either by means of torsion-freeness conditions, as in 5.11, or by imposing the condition that the group concerned should have a bound on its subnormal indices, enabling us to employ Theorem 2.36.

Now we attempt to remove the restriction of p -reducedness from the conditions of Theorem 5.11. Indeed, so long as we retain the p -torsion-freeness requirement, we can obtain surprisingly detailed information about the structure of the group. We need a preliminary lemma.

5.15 LEMMA. *Let $G = \langle A, x \rangle$ where A is an abelian normal subgroup of G and is p -torsion-free for some prime p . Suppose that $[A, x^{p^r}] = 1$ for some non-negative integer r , and that the action of $\langle x \rangle$, when viewed as an automorphism group of A , is fixed-point-free. Then if G has the subnormal intersection property, A is p -radicable and $[A, x] = A$.*

Proof: Let P denote the maximal p -radicable subgroup of A . They by Lemma 3.41, A/P is p -torsion-free, and so the action of $\langle x \rangle$ as an automorphism group of A/P is still fixed-point-free, by Lemma 3.54. On the other hand, since A/P is p -reduced, G/P satisfies all the requirements of Theorem 5.11 and we deduce that G/P is abelian. We must therefore have $A = P$; moreover it follows from Corollary 3.53 that $[A, x] = A$.

5.16 THEOREM. *Let $G = \langle A, x \rangle$ where A is an abelian normal subgroup of G and is p -torsion-free for some prime p . Suppose that $[A, x^{p^r}] = 1$ for some non-negative integer r , and denote by P the maximal p -radicable subgroup of A . Then if G has the*

subnormal intersection property,

$$(i) \quad G' C_P(x) = P, \text{ with } G' \cap C_P(x) = 1;$$

$$(ii) \quad G' = \gamma_3 G;$$

$$(iii) \quad G' C_A(x) = A \text{ with } G' \cap C_A(x) = 1;$$

$$(iv) \quad G' \cap \zeta_1(G) = 1;$$

(v) if a subnormal subgroup S of G contains an element xa , $a \in A$, then $S \geq G'$ and S is normal in G .

Proof: By Lemma 3.55, P is the direct product of $C_P(x)$ and a subgroup D such that $[D, x] = D$. Clearly (i) will be established if we show that $G' \leq D$, that is, that $[A, x] \leq D$. Choose any element a of A . By Theorem 5.11, G/P is abelian, so that $[a, x] \in P$ and $[a, x, x] \in D$. Thus for some element d of D ,

$$[a, x]^{p^r} = [a, x^{p^r}]d = d.$$

Now P/D is p -torsion-free, hence $[a, x] \in D$, proving that $[A, x] \leq D$ as required.

(ii) is now an immediate consequence of the equality $[D, x] = D$, since $D = G'$.

$A/C_A(x)$ is acted on fixed-point-freely by $\langle x \rangle$, from Lemma 3.51. $G/C_A(x)$ inherits the subnormal intersection property, so by Lemma 5.15 we have $[A, x]C_A(x) = A$ or $DC_A(x) = A$. Since $D \cap C_A(x) = D \cap C_P(x) = 1$, (iii) is established.

(iv) is now clear, for

$$G' \cap \zeta_1(G) = D \cap C_G(x) = D \cap C_A(x) = 1.$$

It is evident that for any a in A , $[G', xa] = [G', x] = G'$, so that if a subnormal subgroup S of G contains xa , then for each non-negative integer i ,

$$\gamma_{G'S}^i = G' ,$$

proving that $G' \leq S$ as stated in (v).

To indicate the drastic effect of omitting p -torsion-freeness from the hypotheses of Theorem 5.16, we point out that if $k > 1$, each of the statements of the theorem is false for the group G_k of example 5.14: (i) is false because here $P = 1$ yet G_k is non-abelian; (ii) and (iv) fail because neither can hold in a non-abelian nilpotent group; (iii) is untrue because a cyclic p -group cannot be a direct product of proper subgroups; (v) fails because $\langle ba \rangle$ is a subnormal subgroup of G_k but does not contain G'_k .

However, in spite of the restrictions necessary to obtain any detailed information on the structure of even these simple cases, we will find that we have sufficient tools to deal with much more general situations and to attack some seemingly complex problems.

To complete this section we present an example which shows why we include restrictions of the form $[A', x^k] = 1$ (for some integer k) in the statements of our main theorems. This example is based on techniques used in [26].

5.17 EXAMPLE. Let X be any abelian group and Y an infinite cyclic group. Then the complete wreath product $G = X \text{ Wr } Y$ has the property that every subnormal subgroup of G has defect at most 2.

Proof: Let $Y = \langle y \rangle$ and denote by F the base group of the wreath product. Choose any element f of F and a positive integer r . We show that for some g in F , $f = [g, y^r]$.

Define g (a function from Y into X) as follows:

$$g(y^s) = (a) \quad 1 \quad \text{if } 0 \leq s < r ;$$

$$(b) \quad f(y^s)^{-1} g(y^{s-r}) \quad \text{if } s \geq r ;$$

(c) $f(y^{s+r})g(y^{s+r})$ if $s < 0$.

Then g is recursively well defined, and for any integer $t < r$, (c) yields

$$g(y^{t-r}) = f(y^t)g(y^t).$$

Together with (b), this implies that for any integer t ,

$$g(y^t) = f(y^t)^{-1}g(y^{t-r}).$$

By the standard formula for the action of Y on F , this means that $g = f^{-1}g^{y^r}$, and since F is abelian we have $f = [g, y^r]$ as claimed. Thus $F = [F, y^r]$ for each positive integer r .

Now if S is a subnormal subgroup of G not contained in F , S has an element of the form $y^{r_1}f_1$ for some $f_1 \in F$ and $r_1 > 0$.

Then

$$F = [F, y^{r_1}] = [F, y^{r_1}f_1] \leq [F, S],$$

and by a familiar argument $F \leq S$ follows, showing that S is normal in G . The defect of any subnormal subgroup of G is thus at most 2.

Although the general structure of the group of Example 5.17 may be complex - complex enough to indicate that our theorems fail when we drop the requirement $[A, x^k] = 1$ - its subnormal structure is obviously fairly simple. That this is not in general true of abelian-by-cyclic groups with the subnormal intersection property will become clear in Section 6.3, where we will construct a group of this type which has no bound for its subnormal indices.

5.2 Abelian-by-nilpotent groups

In this section we generalise some of the results of Section 5.1

and investigate the links between the subnormal intersection property and the property of having bounded subnormal indices. First we introduce a convenient piece of terminology.

5.21 DEFINITION. A group will be said to be *factorisable* when it can be expressed as the product of two proper normal subgroups.

We now prove a well-known but useful result.

5.22 LEMMA. *A finite nilpotent group is not factorisable if and only if it is cyclic of prime-power order.*

Proof: Only the necessity of the condition is in doubt. If G is a non-factorisable, finite nilpotent group, G certainly has prime-power order and has a unique maximal proper normal subgroup N . Let $g \in G - N$. Then clearly $\langle g \rangle^G = G$ and, since $\langle g \rangle$ is necessarily subnormal in G , we have $\langle g \rangle = G$.

Our first theorem is a generalisation of 5.12.

5.23 THEOREM. *Let π be a non-empty set of primes, and G a group with an abelian normal subgroup A which is p -reduced and p -torsion-free for each p in π . If G/A is a nilpotent π -group and G has the subnormal intersection property, then A is central in G and G is nilpotent.*

Proof: If $x \in G$ then by the nilpotency of G/A , $\langle A, x \rangle$ is subnormal in G and inherits the subnormal intersection property. Then $\langle A, x \rangle$ satisfies all the conditions of Theorem 5.12, including π -torsion-freeness of A so that $\langle A, x \rangle$ is abelian. Since x was an arbitrary element of G , A is central in G and G is nilpotent.

We point out that the requirement of p -torsion-freeness, which amounts to torsion-freeness and becomes redundant if π contains more

than one prime, is certainly not redundant in the case $\pi = \{p\}$, in view of the group discussed in 2.44. We can, however, regain the conclusion of the theorem when A is not necessarily p -torsion-free, at the expense of a restriction on G/A .

5.24 THEOREM. *Let p be a prime and G a group with a p -reduced abelian normal subgroup A such that G/A is a finite p -group. If G has the subnormal intersection property then G is nilpotent.*

Proof: We proceed by induction on $|G/A|$, noting that if $|G/A| = 1$ the result is trivially true. Suppose that $|G/A| = k$ and that the result holds for all pairs (H, B) with $|H/B| < k$, where $k > 1$. If G/A is factorisable then $G = G_1 G_2$ where $|G_1/A| < k$ and $|G_2/A| < k$. G_1 and G_2 satisfy the hypotheses of the theorem, so by the induction hypothesis G_1 and G_2 are nilpotent, and by Fitting's Theorem G is nilpotent. But if G/A is not factorisable then, since it is finite and nilpotent, it must be cyclic, by Lemma 5.22, and the nilpotency of G follows from Theorem 5.11. Thus the result holds for $|G/A| = k$ and our inductive proof is complete.

We remark that in view of examples 5.13 and 2.44, both the p -reducedness of A and the finiteness of G/A are essential in the statement of the theorem.

An alternative method of removing the condition of torsion-freeness in 5.23 is to insist that the group G have a bound on its subnormal indices.

5.25 THEOREM. *Let p be a prime and G a group with a p -reduced abelian normal subgroup A such that G/A is a nilpotent p -group. If G has a bound on its subnormal indices then G is nilpotent.*

Proof: Suppose n is the bound for the subnormal indices in G . Write $m = R(n) + 1$, where R is the function of Theorem 2.36. Choose any elements g_1, \dots, g_m of G and consider the subgroup $H = \langle A, g_1, \dots, g_m \rangle$. Since G/A is nilpotent, H is subnormal in G and so H inherits the bound n for its subnormal indices. Moreover H/A is finite since it is a finitely-generated, nilpotent p -group. Thus H satisfies the conditions of Theorem 5.24 and is therefore nilpotent. But then by Theorem 2.36 the nilpotent class of H is at most $R(n)$ and so $[g_1, \dots, g_m] = 1$. Since g_1, \dots, g_m were arbitrary elements, this proves that G is nilpotent of class at most $R(n)$.

We remark that Examples 5.13 and 2.44 show the necessity of the conditions concerning the p -reducedness of A and the bound on the subnormal indices.

To conclude this section we prove two related theorems linking the subnormal intersection property with the property of having bounded subnormal indices.

5.26 THEOREM. *Let G be a group with a normal abelian subgroup A such that G/A is a finite nilpotent group. Then G has the subnormal intersection property if and only if G has a bound on its subnormal indices.*

Proof: To prove the non-trivial half of the theorem, suppose G has the subnormal intersection property and let S be any subnormal subgroup of G . Since $S/S \cap A$ is isomorphic to a subgroup of G/A , we can express $S/S \cap A$ as the direct product of its Sylow subgroups $\{S_p/S \cap A : p \in \pi\}$ where π is the set of primes dividing $|G/A|$.

For the moment let p be a fixed prime in π and let P denote the maximal p -radicable subgroup of A . Now $S_p \cap A$ is a normal

subgroup of $S_p P$. In the group $S_p P / (S_p \cap A)$, $S_p / (S_p \cap A)$ is a subnormal p -subgroup and $P(S_p \cap A) / (S_p \cap A)$ is a p -radicable abelian normal subgroup. Thus we can apply Lemma 3.46 to deduce that $S_p / (S_p \cap A)$ is normal in $S_p P / (S_p \cap A)$ or that S_p is normal in $S_p P$. If we write G_p/A for the Sylow p -subgroup of G/A , then G_p is normal in G and G_p/P satisfies the conditions of Theorem 5.24, so G_p/P is nilpotent, of class k_p , say. It follows that the subnormal index of S_p in G is at most $k_p + 2$, since $S_p P \leq G_p$.

If we now let p range freely over π , the subnormal index of S_p in G is at most $k + 2$, where $k = \max\{k_p : p \in \pi\}$.

Now $SA/(S \cap A)$, being the join of an abelian normal subgroup $A/(S \cap A)$ and a nilpotent subnormal subgroup $S/(S \cap A)$, is a nilpotent group, by Lemma 2.22. Then $S/(S \cap A)$ lies in the torsion subgroup $T/(S \cap A)$ of $SA/(S \cap A)$; moreover each term of the standard series of $S/(S \cap A)$ in $T/(S \cap A)$ is just the direct product of the corresponding terms of the standard series of $S_p/(S \cap A)$ as p ranges over π . Hence $s(T/(S \cap A) : S/(S \cap A)) \leq k + 2$. If c is the nilpotent class of G/A , we then have

$$s(G : S) \leq s(G : SA) + s(SA : T) + s(T : S)$$

$$\leq c + 1 + (k+2) = k + c + 3.$$

Since S was an arbitrary subnormal subgroup of G and the integer $(k+c+3)$ is independent of the choice of S , the proof of the theorem is complete.

As usual, we can relax our requirements on G/A if we impose a restriction on A . Let π be a set of primes with more than one

5.27 THEOREM. *Let π be a non-empty set of primes and G a group with a π -torsion-free abelian normal subgroup A such that G/A is a nilpotent π -group. Then G has the subnormal intersection property if and only if G has a bound on its subnormal indices.*

Proof: To prove the non-trivial half of the theorem, suppose that G has the subnormal intersection property. Let p be a fixed prime in π and define G_p and P as in Theorem 5.26. Then by Lemma 3.41, A/P is p -torsion-free, so that G_p/P now satisfies the conditions of Theorem 5.23. Then A/P is central in G_p/P ; if c is the nilpotent class of G/A then G_p/P is nilpotent of class at most $c + 1$.

It is not difficult to see that we can now apply almost verbatim the proof of Theorem 5.26, taking $k = c + 1$. We deduce that $2c + 4$ is a bound for the subnormal indices in G , which proves the result.

(Note the interesting fact that the bound depends only on the class of G/A .)

5.3 Metanilpotent groups

We now extend the techniques of the previous two sections to the more general context of metanilpotent groups. Our first result appears as Lemma 3.5 in the attached paper [A], to which we refer for the proof.

5.31 THEOREM. *Let p be a prime and G a group with a p -reduced nilpotent normal subgroup N such that G/N is a cyclic p -group. If G has the subnormal intersection property, G is nilpotent.*

We now prove a result which to some extent generalises 5.31.

5.32 THEOREM. *Let π be a set of primes with more than one*

member. Suppose a group G has a nilpotent normal subgroup N which is p -reduced for each p in π and G/N is a nilpotent π -group. If G has the subnormal intersection property then G is nilpotent of class at most the sum of the classes of N and G/N .

Proof: If $N = 1$ there is nothing to prove, so let $N > 1$ and write A for $\zeta_1(N)$. Then N/A is p -reduced for each prime p in π , by Lemma 3.44. Since G/A inherits the subnormal intersection property we may suppose, for a proof by induction on n , the nilpotent class of N , that G/A is nilpotent of class at most $m + n - 1$, where m is the class of G/N . If $x \in G$ then by the induction hypothesis $\langle x, A \rangle$ is subnormal in G and inherits the subnormal intersection property. For some π -number k , $x^k \in N$ and $[x^k, A] = 1$. Moreover, since π has more than one member, the conditions on N imply that N and A are torsion-free. Thus the group $\langle x, A \rangle$ satisfies all the requirements of Theorem 5.12 and so is abelian. It follows that A is central in G and that G is nilpotent of class at most $m + n$, completing our inductive proof.

The well-worn example 2.44 shows that for $\pi = \{p\}$ the theorem fails. However, we can impose further restrictions to make it work.

5.33 THEOREM. Let p be a prime and G a group with a p -reduced nilpotent normal subgroup N such that G/N is a finite p -group. If G has the subnormal intersection property then G is nilpotent.

Proof: We proceed by induction on $n = |G/N|$, noting that the case $G = N$ is trivial. Suppose that $n > 1$, and that the result holds for groups in which the order of the relevant factor group is less than n .

If G/N is not factorisable then by Lemma 5.22 it is cyclic;

thus by Theorem 5.31, G is nilpotent. We may therefore assume that G is expressible as the product of two normal subgroups G_1 and G_2 , each containing N , with $|G_1/N|$ and $|G_2/N|$ both less than n . Then by the induction hypothesis G_1 and G_2 are nilpotent, and Fitting's Theorem yields the nilpotency of G , completing the proof.

As usual, we can replace the condition that G/N be finite by a restriction on N .

5.34 THEOREM. *Let p be a prime and G a group with a nilpotent normal subgroup N which is p -reduced and p -torsion-free. If G/N is a nilpotent p -group and G has the subnormal intersection property then G is nilpotent of class at most the sum of the classes of N and G/N .*

Proof: If $N = 1$ there is nothing to prove, so let $N > 1$ and put $A = \zeta_1(N)$. Then N/A is p -reduced by Lemma 3.44, and is p -torsion-free by Corollary 3.22. Since G/A inherits the subnormal intersection property we may suppose, for a proof by induction on the nilpotent class n of N , that G/A is nilpotent of class at most $m + n - 1$, where m is the class of G/N . Then the argument of Theorem 5.32, using 5.12, shows that A is in fact central in G . Hence G is nilpotent of class at most $m + n$, as required.

We give another variant of Theorem 5.32: the proof will be omitted since it follows the lines of Theorem 5.25, writing N for A and using 5.33 in place of 5.24.

5.35 THEOREM. *Let p be a prime and G a group with a p -reduced nilpotent normal subgroup N such that G/N is a nilpotent p -group. If G has a bound on its subnormal indices then G is nilpotent.*

We point out that in view of Example 5.14 it is clear that the nilpotent classes of the groups G of 5.33 and 5.35 cannot be bounded as in 5.34.

Like the previous one, this section ends with some results linking the subnormal intersection property and the property of having bounded subnormal indices. We need a result which appears as Lemma 3.7 in the attached paper [A], where the proof, due to D. McDougall, may be found.

5.36 LEMMA. *Let π be a non-empty set of primes and G a group with a nilpotent normal subgroup N such that G/N is a π -group. Let $Q(\pi)$ be the maximal π -radicable subgroup of N . If N has nilpotent class c and S is any subnormal subgroup of G , then $s(SQ(\pi) : S) \leq c$.*

5.37 THEOREM. *Let G be a group with a nilpotent normal subgroup N such that G/N is finite and nilpotent. Then G has the subnormal intersection property if and only if G has a bound on its subnormal indices.*

Proof: We prove the non-trivial half of the theorem by induction on $n = |G/N|$, noting that the case $G = N$ is trivial. Suppose then that $n > 1$, and that the result holds for groups in which the order of the relevant factor group is less than n .

If G/N is not factorisable then by Lemma 5.22, G/N is a cyclic p -group for some prime p . Then denoting by P the maximal p -radicable subgroup of N , Theorem 5.33 implies that G/P is nilpotent, say of class d . If the nilpotent class of N is c and S is any subnormal subgroup of G then $s(G : S) \leq s(G : SP) + s(SP : S) \leq d + c$, using Lemma 5.36. Thus in this case the subnormal indices are bounded.

We may therefore assume that G/N is factorisable. Now if S

is a subnormal subgroup of G with $SN < G$, then SN coincides with one of the finite set of proper subnormal subgroups of G which contain N . Each subgroup in this set has a bound on its subnormal indices, by the induction hypothesis, thus there is an integer k such that $s(SN : S) \leq k$, where k is independent of S . If the nilpotent class of G/N is m , we then have $s(G : S) \leq k + m$. On the other hand if S is a subnormal subgroup of G with $SN = G$, then by the factorisability of G/N , S can be expressed as the product of two subgroups S_1 and S_2 , each normal in S , with S_1N/N and S_2N/N proper normal subgroups of G/N . Then S_1 and S_2 are subnormal in G with subnormal indices at most $k + m$, by the remarks above. Hence, by Lemma 2.53, $s(G : S) \leq (k+m)^2$. To sum up, any subnormal subgroup S of G will have $s(G : S) \leq (k+m)^2$; this latter integer is independent of S , so our inductive proof is complete.

Our final theorem in this section is a predictable variant of 5.37.

5.38 THEOREM. *Let π be a non-empty set of primes and G a group with a π -torsion-free nilpotent normal subgroup such that G/N is a nilpotent π -group. Then G has the subnormal intersection property if and only if G has a bound on its subnormal indices.*

Proof: To prove the non-trivial half of the theorem we argue by induction on the nilpotent class n of N , noting that the case $N = 1$ is trivial. We therefore assume that G has the subnormal intersection property and that G/A has a bound on its subnormal indices, where $A = \zeta_1(N)$. For each p in π let G_p/N denote the Sylow p -subgroup of G/N , and P the maximal p -radicable subgroup of N . If S is any subnormal subgroup of G , let $S_p/S \cap N$

denote the Sylow p -subgroup of $S/S \cap N$. Then S_p is a subnormal subgroup of G_p , and by Lemma 5.36 we have $s(S_p^P : S_p) \leq n$. Now by Theorem 5.34, G_p/P is nilpotent of class at most $m + n$, where m is the class of G/N . Thus $s(G : S_p) \leq m + 2n + 1$, for each p in π .

We can now apply the argument of 5.26 to the group $SA/S \cap N$, to deduce that $s(SA : s) \leq m + 2n + 2$, giving a bound for $s(G : S)$ and completing the proof.

5.4 Abelian-by-finite groups

In this final section of the chapter we show that an abelian-by-finite group with the subnormal intersection property has a bound on its subnormal indices. As a preliminary we need some additional terminology.

Let G be a group with a series

$$(*) \quad \{\bigwedge_{\sigma}, \bigvee_{\sigma} : \sigma \in \Omega\}$$

for some linearly ordered set Ω (see Section 3.1). Let A be the automorphism group of G . Then to each factor $\bigwedge_{\sigma} \bigvee_{\sigma}$ of the series

(*) there corresponds a subgroup $C_{\sigma} = C_A(\bigwedge_{\sigma} \bigvee_{\sigma})$ of A , consisting

of those automorphisms α of G for which $[x, \alpha]$ and $[x, \alpha^{-1}]$

lie in \bigvee_{σ} for each x in \bigwedge_{σ} . That these automorphisms form a

group is a consequence of the validity of the commutator identity

2.13 (ii)(a) in the holomorph of G , giving, for $x \in \bigvee_{\sigma}$,

$$\alpha, \beta \in C_{\sigma},$$

$$[x, \alpha\beta] = [x, \beta][x, \alpha][x, \alpha, \beta].$$

5.41 DEFINITION ([12]). We define the *stability group* Γ of the series (*) to be the group

$$\cap \{C_\sigma : \sigma \in \Omega\} = \cap \{C_A(\bigwedge \bigvee_\sigma) : \sigma \in \Omega\}.$$

5.42 DEFINITION ([12]). We say that a group H can faithfully stabilise the series (*) of G if there is a monomorphism of H into the stability group Γ of (*).

We now state, without proof, a result of Hall and Hartley.

5.43 THEOREM ([12], Lemma 16). A periodic group H which can faithfully stabilise an invariant descending series of some group has an abelian normal subgroup M , such that in the group H/M elements of coprime orders commute.

Since any finite group in which elements of coprime orders commute is necessarily nilpotent, we have an immediate corollary.

5.44 COROLLARY. A finite group H which can faithfully stabilise an invariant descending series of some group has an abelian normal subgroup M such that H/M is nilpotent.

Before proceeding to the main theorem of this section, we state a well-known result, due to Schur, which can be found, for example, in [9] (Theorem 8.1).

5.45 LEMMA. If G is a group such that $G/\zeta_1(G)$ is finite, then G' is finite.

We are now in a position to prove the promised result.

5.46 THEOREM. Let G be a group with an abelian normal subgroup A such that G/A is finite. Then G has the subnormal intersection property if and only if G has a bound on its subnormal indices.

Proof: To prove the non-trivial half of the theorem, we argue by contradiction. Let G be a counterexample in which the factor group

G/A is of least possible order (clearly $A < G$), that is, G is a group with the subnormal intersection property but with unbounded subnormal indices. We show that we can make the following assumptions on G :

- (i) $\cap \{\gamma_A G^i : i \geq 0\} = 1$;
- (ii) $A = C_G(A)$.

(i) Every member H of the finite set of proper subnormal subgroups of G containing A will satisfy the hypotheses of the theorem, with $|H/A| < |G/A|$. By the minimality of $|G/A|$ it is then possible to find a positive integer k , independent of H , which will bound the subnormal indices (in H) of subnormal subgroups of H . If d is the composition length of G/A , then for any subnormal subgroup S of G such that $SA < G$ we must have

$$s(G : S) \leq s(G : SA) + s(SA : S) \leq d + k .$$

If we now denote by Σ the set of subnormal subgroups S of G with $SA = G$, the remarks above indicate that the subnormal indices of the subgroups in Σ must be unbounded, since G is a counterexample. But if $S \in \Sigma$ then for some positive integer m , depending on S , we have

$$\gamma_A G^m = \gamma_A (SA)^m = \gamma_A S^m \leq S .$$

Hence S certainly contains $B = \cap \{\gamma_A G^i : i \geq 0\}$ and it is clear that G/B is also a counterexample to the theorem. Thus we may assume that G satisfies (i), by replacing G by G/B .

- (ii) Now suppose that in this counterexample $A < C = C_G(A)$.

Then C and C' are normal subgroups of G , and G/C' has an abelian normal subgroup C/C' , with $|G/C| < |G/A|$. Since G/C' satisfies the hypotheses of the theorem, the minimality of $|G/A|$ implies that G/C' has a bound on its subnormal indices. But

$A \leq \zeta_1(C)$, so that by Lemma 5.45, C' is finite. Then by Lemma 2.34, G has a bound on its subnormal indices, contradicting the choice of G . We have thus shown that $A = C_G(A)$, and we may assume that G satisfies both (i) and (ii).

Now by (i) the family of subgroups

$$\{\gamma A G^i : 0 \leq i \leq \omega\}$$

(where ω denotes the first infinite ordinal) forms an invariant descending series of A . There is clearly a homomorphism from G/A into the stability group of this series; moreover, by (ii), this homomorphism has trivial kernel and so G/A can faithfully stabilise the series. By 5.44 there is a normal subgroup M of G , containing A , such that M/A is abelian and G/M nilpotent.

Now by Theorem 5.26, M has a bound on its subnormal indices. Then by Corollary 2.37 the lower central series of M terminates after finitely many steps at M_1 , say. Then M_1 is a normal subgroup of G , $M_1 \leq A$ and $[M_1, M] = M_1$. A simple inductive argument shows that for each non-negative integer i ,

$$M_1 \leq \gamma A G^i.$$

It then follows from (i) that $M_1 = 1$ and thus M is nilpotent. But since G/M is nilpotent we may apply Theorem 5.37 to deduce that G has a bound on its subnormal indices. With this contradiction to our choice of G the theorem is proved.

It would be interesting to know whether a result analogous to 5.46 holds for nilpotent-by-finite groups. Theorem 5.37 indicates that it should, and in 6.21 we will prove it in a restricted setting. However, a general proof along the lines of Theorem 5.46 will not be easy, as the conditions (i) and (ii) are more difficult to handle in the nilpotent-by-finite case.

CHAPTER 6

SUBNORMAL STRUCTURE IN SOLUBLE GROUPS OF FINITE RANK

In this chapter we continue our investigations of the restrictions under which a group with the subnormal intersection property has a bound on its subnormal indices. Motivated by the fact that our standard example 2.44 does not have finite p -rank, we attempt to decide what rank restrictions on soluble groups will ensure that the two properties coincide.

The major part of Section 6.1 is devoted to the rather long proof of the essential Theorem 6.12; there is some discussion of analogous results involving the concept introduced in Definition 6.11. In Section 6.2 we show how Theorem 6.12, together with a result of McDougall, can be used to show that any soluble minimax group with the subnormal intersection property has a bound on its subnormal indices. We construct in Section 6.3 an example to show that the corresponding statement is not true if we insist only on finite reduced rank instead of minimax, even for abelian-by-cyclic groups.

6.1 A useful theorem

In this section we prove an important theorem, using material from Chapters 2, 3 and 5. In order to state the theorem we need the following definition.

6.11 DEFINITION. Let G be a group with a normal subgroup N . Denote by $\Sigma(G : N)$ the family of subnormal subgroups S of G such that $SN = G$; clearly $\Sigma(G : N)$ is non-empty. Define a subgroup $\sigma(G : N)$ of G by

$$\sigma(G : N) = \bigcap \{S : S \in \Sigma(G : N)\}.$$

We make the following observations on this definition.

(a) If N is a normal subgroup of G and θ an inner

automorphism of G , then for any $S \in \Sigma(G : N)$, it is easy to see that $S\theta^{-1} \in \Sigma(G : N)$. Thus for any $g \in \sigma(G : N)$, $g \in S\theta^{-1}$, or $g\theta \in S$. This holds for any $S \in \Sigma(G : N)$ so that $g\theta \in \sigma(G : N)$ follows. Hence $\sigma(G : N)$ is invariant under θ , and is thus a normal subgroup of G , since θ was an arbitrary inner automorphism of G . Moreover, if N happens to be a characteristic subgroup of G , the same argument, deleting the work "inner", shows that $\sigma(G : N)$ is then a characteristic subgroup of G .

(b) If N is a normal subgroup of G and M a subgroup of G containing N , any subgroup S in $\Sigma(G : N)$ will have the property that $S \cap M$ is subnormal in M , and $(S \cap M)N = M \cap SN = M$, using the modular law. Thus $S \cap M \in \Sigma(M : N)$, and so $\sigma(M : N)$ is a subgroup of $\sigma(G : N)$.

We are now in a position to state and prove the promised theorem.

6.12 THEOREM. *Let p be a prime, and G a group with a p -torsion-free, nilpotent normal subgroup N such that G/N is a finite p -group. If $\sigma(G : N) = 1$ and G has the subnormal intersection property, then G is nilpotent.*

Proof: Let P denote the maximal p -radicable subgroup of N . Let us define a chain $\{P_i : i \geq 0\}$ of normal subgroups of G as follows: $P_0 = 1$; P_i/P_{i-1} is the maximal p -radicable subgroup of $\zeta_1(N/P_{i-1})$, for $i \geq 1$. Applying Lemma 3.42 we see that P_i is precisely the maximal p -radicable subgroup of $\zeta_i(N)$ and that $P_m = P_{m+1} = P$, where m is the nilpotent class of N . Moreover, since the group $P/P \cap \zeta_i(N)$ is isomorphic to $P\zeta_i(N)/\zeta_i(N)$, which is p -torsion-free by Corollary 3.22, it follows that $P \cap \zeta_i(N)$ is p -radicable; hence P_i coincides with $P \cap \zeta_i(N)$, and P/P_i is

p -torsion-free. To sum up, then, for each i with $0 < i \leq m$, P_i/P_{i-1} is a p -radicable, p -torsion-free abelian group which is central in N/P_{i-1} .

If, for $0 < i \leq m$, we denote by C_i the centraliser in G of the factor P_i/P_{i-1} , our last statement implies that $N \leq C_i$. Thus the action of G on P_i/P_{i-1} is essentially that of the finite p -group G/C_i , and we are in a position to apply Lemma 3.55. We deduce that P_i is the product of two subgroups K_i and L_i , each normal in G , with the following properties:

- (i) $P_{i-1} \leq K_i \cap L_i$;
- (ii) $[L_i, G] \leq P_{i-1}$;
- (iii) $K_i = [P_i, G]P_{i-1} = [K_i, G]P_{i-1}$.

The equations hold for $0 < i \leq m$. If we put $K_0 = L_0 = 1$, it is easy to see from (iii) that $[K_i, G] \geq [K_{i-1}, G]$, and that (iii) can then be extended to:

- (iv) $K_i = [K_i, G]L_{i-1} = \gamma_{K_i, G}^r.L_{i-1}$, for any integer $r \geq 0$.

Finally, the centrality of P_i/P_{i-1} in N/P_{i-1} implies that for any S in $\Sigma(G : N)$, we have, substituting SN for G in (iv), that

- (v) $K_i = [K_i, S]L_{i-1} = \gamma_{K_i, S}^r.L_{i-1}$ for any integer $r \geq 0$.

Since we know from Theorem 5.33 that G/P is nilpotent, our objective is to prove that for some integer k , $\gamma_{PG}^k = 1$. We accomplish this by showing that for each i with $0 < i \leq m$, $K_i = P_{i-1}$, that is, $P_i = L_i$ (which will show that in fact $\gamma_{PG}^m = 1$).

(a) Suppose that this statement is false, and let K_t be the first member of the chain $\{K_i : i > 0\}$ which violates it. Then

$K_t > P_{t-1}$. Firstly we note that if S is an arbitrary subgroup in $\Sigma(G : N)$, $\gamma_{K_1} S^r = K_1$, for any $r \geq 0$, by (v) above. From the subnormality of S it then follows that $K_1 \leq S$, and thus that $K_1 \leq \sigma(G : N) = 1$. This shows that $t > 1$.

The minimality of t implies that for any i with $0 < i < t$, $P_i = L_i$ and $[P_i, G] \leq P_{i-1}$. An obvious inductive argument yields

$$(a) \quad \gamma_{P_{t-1}} G^{t-1} = \gamma_{L_{t-1}} G^{t-1} = 1.$$

Since L_{t-1} is normal in G , we may apply Lemma 2.17 to deduce that

$$(b) \quad [L_{t-1}, \gamma_{t-1} G] = 1.$$

Now we proceed to establish by induction on i that for $0 < i \leq m$,

$$(c) \quad [P_i \cap \gamma_{t-1} G, K_t] = 1.$$

Consider the subgroup $P_{t-1} \cap \gamma_{t-1} G = L_{t-1} \cap \gamma_{t-1} G$. We can write

$$K_t = \gamma_{K_t} G^r . L_{t-1}$$

for any non-negative integer r , by (iv). Thus

$$[P_{t-1} \cap \gamma_{t-1} G, K_t] = [L_{t-1} \cap \gamma_{t-1} G, K_t G^{t-1} . L_{t-1}].$$

Applying (b) twice, we see that

$$[P_{t-1} \cap \gamma_{t-1} G, K_t] = 1,$$

so that (c) holds for $0 < i \leq t-1$, at least.

Suppose now that (c) holds with $i = j < m$. We show first of all that

$$(d) \quad [K_{j+1} \cap \gamma_{t-1}G, K_t] = 1 .$$

Let us denote by M_{j+1} the commutator which appears in (d). If S is any subgroup in $\Sigma(G : N)$, say of defect s in G , we may choose an integer $r \geq s + t$ and write, using (v),

$$M_{j+1} = [K_{j+1} \cap \gamma_{t-1}G, K_t] = \left[\left(\gamma_{K_{j+1}} S^r \cdot L_j \right) \cap \gamma_{t-1}G, K_t \right] .$$

By the choice of r ,

$$\gamma_{K_{j+1}} S^r \leq \gamma_{t-1}G ,$$

and so we have, by the modular law,

$$M_{j+1} = \left[(L_j \cap \gamma_{t-1}G) \cdot \gamma_{K_{j+1}} S^r, K_t \right] .$$

But by our induction hypothesis,

$$[L_j \cap \gamma_{t-1}G, K_t] = 1 ,$$

so we have

$$M_{j+1} = \left[\gamma_{K_{j+1}} S^r, K_t \right] .$$

Now, using (v) again, we may write

$$M_{j+1} = \left[\gamma_{K_{j+1}} S^r, L_{t-1} \gamma_{K_t} S^r \right]$$

which, recalling the choice of r and (b) above, becomes

$$M_{j+1} = \left[\gamma_{K_{j+1}} S^r, \gamma_{K_t} S^r \right] .$$

Since r exceeds the defect s of S , we have shown that $M_{j+1} \leq S$.

It follows that, since S was an arbitrary member of $\Sigma(G : N)$,

$$M_{j+1} = [K_{j+1} \cap \gamma_{t-1}G, K_t] \leq \sigma(G : N) = 1 ,$$

proving (d).

Let us denote by R_{j+1} the commutator

$$[P_{j+1} \cap \gamma_{t-1}G, K_t] = [K_t, P_{j+1} \cap \gamma_{t-1}G] .$$

Using (iv) we may write

$$R_{j+1} = [L_{t-1}[K_t, \bar{G}], P_{j+1} \cap \gamma_{t-1}G]$$

which, in view of (b), becomes

$$R_{j+1} = [K_t, G, P_{j+1} \cap \gamma_{t-1}G] .$$

Applying the "three subgroup lemma" 2.15, we deduce that

$$R_{j+1} \leq [G, P_{j+1} \cap \gamma_{t-1}G, K_t][P_{j+1} \cap \gamma_{t-1}G, K_t, \bar{G}] .$$

Now by (iii) we have

$$[G, P_{j+1} \cap \gamma_{t-1}G, K_t] \leq [K_{j+1} \cap \gamma_{t-1}G, K_t] ,$$

and so by (d) above

$$(e) \quad [G, P_{j+1} \cap \gamma_{t-1}G, K_t] = 1 .$$

It then follows that

$$(f) \quad R_{j+1} \leq [P_{j+1} \cap \gamma_{t-1}G, K_t, \bar{G}] = [R_{j+1}, \bar{G}] .$$

But now

$$\begin{aligned} [R_{j+1}, N] &= [P_{j+1} \cap \gamma_{t-1}G, K_t, N] \\ &\leq [K_t, N, P_{j+1} \cap \gamma_{t-1}G] \cdot [N, P_{j+1} \cap \gamma_{t-1}G, K_t] , \end{aligned}$$

by Lemma 2.15. The second of these commutators is trivial, by (e),

hence

$$\begin{aligned} [R_{j+1}, N] &\leq [K_t, N, P_{j+1} \cap \gamma_{t-1}G] \\ &\leq [P_{t-1}, P_{j+1} \cap \gamma_{t-1}G] = [L_{t-1}, P_{j+1} \cap \gamma_{t-1}G] , \end{aligned}$$

which is trivial, by (b) above.

Since $[R_{j+1}, N] = 1$, it follows from (f) above that for any subgroup S in $\Sigma(G : N)$,

$$R_{j+1} \leq [R_{j+1}, G] = [R_{j+1}, NS] = [R_{j+1}, S] .$$

It is then clear that for any $r \geq 0$,

$$\gamma^{R_{j+1}} S^r = R_{j+1} .$$

Since S is subnormal in G , we must have $R_{j+1} \leq S$, and so

$R_{j+1} \leq \sigma(G : N) = 1$. This proves that

$$[P_{j+1} \cap \gamma_{t-1} G, K_t] = 1 ,$$

showing that (c) holds for each i with $0 < i \leq m$, as claimed.

Before proceeding with the proof of the theorem, we digress to prove an essential equality. Let Y be a normal subgroup of G which is contained in N , and denote by $Y_{(i)}$ the subgroup

$$[\gamma Y G^i, K_t] , \text{ for each } i \geq 0 . \text{ Then}$$

$$(g) \quad \gamma_{Y(0)} N^{t-2} = \gamma_{Y(i)} N^{t-2} \quad \text{for each } i \geq 0 .$$

Since we know that $t > 1$, these commutators are defined. We will establish (g) by showing that for $i \geq 0$,

$$\gamma_{Y(i)} N^{t-2} = \gamma_{Y(i+1)} N^{t-2} .$$

Now, for each $i \geq 0$,

$$\begin{aligned} Y_{(i)} &= [K_t, \gamma Y G^i] \\ &= [[K_t, G]_{L_{t-1}}, \gamma Y G^i] , \text{ by (iv).} \end{aligned}$$

From this we obtain

$$Y_{(i)} \leq [K_t, G, \gamma Y G^i] [L_{t-1}, G] .$$

Using the "three subgroup lemma" 2.15 we then have

$$Y_{(i)} \leq [G, \gamma Y G^i, K_t] [\gamma Y G^i, K_t, G] [L_{t-1}, G] ,$$

since all the subgroups involved are normal in G .

Now, since $Y \leq N$,

$$[\gamma Y G^i, K_t] \leq P_{t-1} = L_{t-1} .$$

It then follows that

$$Y_{(i)} \leq [\gamma Y G^{i+1}, K_t] [L_{t-1}, G]$$

or

$$Y_{(i)} \leq Y_{(i+1)} [L_{t-1}, G] .$$

Then it is clear that

$$\gamma_{(i)}^{Y_{(i)}} N^{t-2} \leq \gamma_{(i+1)}^{Y_{(i+1)}} N^{t-2} \cdot \gamma [L_{t-1}, G] N^{t-2} .$$

Since, by (a),

$$\gamma_{L_{t-1}}^{G^{t-1}} = 1 ,$$

we must have

$$\gamma_{(i)}^{Y_{(i)}} N^{t-2} \leq \gamma_{(i+1)}^{Y_{(i+1)}} N^{t-2} .$$

The reverse inclusion being trivial, we have established the desired equality, and with it (g).

Returning to the main stream of the proof, we now show by induction on i that, for each i with $0 < i \leq m$,

$$(h) \quad \gamma [P_i, K_t] N^{t-2} = 1 .$$

From a consideration of equality (a) above, it is clear that (h) is valid for $0 < i \leq t-1$. Let j be an integer with $t-1 \leq j < m$, and suppose that

$$\gamma [P_j, K_t] N^{t-2} = 1 .$$

We seek to show that

$$\gamma [P_{j+1}, K_t] N^{t-2} = 1 ,$$

but, since $[P_{j+1}, G] \leq K_{j+1}$ by (iii), we see that, in view of (g),

we need only establish the equality

$$\gamma [K_{j+1}, K_t] N^{t-2} = 1 .$$

But by (iv) we may write

$$\begin{aligned} \gamma [K_{j+1}, K_t] N^{t-2} &= \gamma \left(\left[\gamma_{K_{j+1}}^{G^t \cdot L_j}, K_t \right] \right) N^{t-2} \\ &= \gamma \left(\left[\gamma_{K_{j+1}}^{G^t}, K_t \right] \right) N^{t-2} \cdot \gamma [L_j, K_t] N^{t-2} . \end{aligned}$$

By the induction hypothesis, this last commutator is trivial, so

$$\begin{aligned}\gamma[K_{j+1}, K_t]N^{t-2} &= \gamma\left[\gamma K_{j+1}^{G^t}, K_t\right]N^{t-2} \\ &\leq \gamma[P_{j+1} \cap \gamma_{t-1}G, K_t]N^{t-2}\end{aligned}$$

which is trivial, by (c). Hence

$$\gamma[P_{j+1}, K_t]N^{t-2} = 1 ,$$

and the proof of (h) is complete.

Taking $i = m$ in (h) yields

$$\gamma[P_m, K_t]N^{t-2} = \gamma[P, K_t]N^{t-2} = 1 .$$

Since G/P has been seen to be nilpotent, by Theorem 5.33, there is a non-negative integer r such that $\gamma N G^r \leq P$. Then, recalling the notation of (g) above, we have

$$[\gamma N G^r, K_t] = N_{(r)} \leq P_{(0)} = [P, K_t] .$$

Since $\gamma P_{(0)}N^{t-2} = 1$, we then have, by (g), with $Y = N$

$$\gamma N_{(0)}N^{t-2} = \gamma N_{(r)}N^{t-2} \leq \gamma P_{(0)}N^{t-2} = 1 .$$

In other words

$$1 = \gamma[N, K_t]N^{t-2} = \gamma K_t N^{t-1} .$$

But if this were true we would have

$$K_t \leq \zeta_{t-1}(N) ,$$

and

$$K_t \leq P \cap \zeta_{t-1}(N) = P_{t-1} ,$$

contradicting our original assumption on K_t . By remarks at the

beginning of the proof, this contradiction establishes that

$\gamma P G^m = 1$, and since G/P is nilpotent, we deduce that G is

nilpotent, completing the proof.

By using some earlier results we can prove a considerably more general theorem than 6.12.

6.13 THEOREM. *Let π be a non-empty set of primes, and G a group with a π -torsion-free, nilpotent normal subgroup N such that G/N is a nilpotent π -group. If $\sigma(G : N) = 1$ and G has the subnormal intersection property, then G is nilpotent.*

Proof. By Theorem 5.38, G has a bound k on its subnormal indices. Let $m = R(k) + 1$, where R is the function of Theorem 2.36, and choose any m elements g_1, \dots, g_m of G . Write

$$H = \langle N, g_1, \dots, g_m \rangle.$$

H/N is a finite nilpotent group and can be expressed as the product of its Sylow subgroups $\{H_p/N : p \in \pi\}$. Consider one of these subgroups H_p . H_p is subnormal in G and thus inherits the bound k for its subnormal indices. Note also that by remark (b) after 6.11, $\sigma(H_p : N) = 1$. H_p therefore satisfies the conditions of Theorem 6.12, and so is nilpotent. H is therefore nilpotent by Fitting's Theorem, and its nilpotency class is at most $R(k)$, by Theorem 2.36. It follows that

$$[g_1, \dots, g_m] = 1,$$

showing that G is nilpotent of class at most $R(k)$.

It may be of interest to consider whether the conditions of Theorem 6.12 can be varied without detriment to the conclusion.

(a) The group G of Example 5.13 is an extension by a 2-cycle $\langle x \rangle$ of a torsion-free abelian group A (all rationals with denominators a power of 2). We showed that every proper subnormal subgroup of G is normal in G and is contained in A . Hence $\sigma(G : A) = G$. Thus G satisfies all the conditions of Theorem 6.12,

with the exception of $\sigma(G : A) = 1$, and yet, as noted in 5.13, G is not nilpotent.

(b) The infinite dihedral group D of Example 2.35 is an extension of an infinite cyclic group A by a 2-cycle B . Moreover, since, in the notation of 2.35, $D_n A = D$ for each $n \geq 0$, and D_n is subnormal in D , it follows that $\sigma(D : A) \leq \bigcap \{D_n : n \geq 0\} = B$. Since B is not normal in D we must have $\sigma(D : A) = 1$. The only condition, therefore, of Theorem 6.12 which D does not satisfy is the possession of the subnormal intersection property. But D is not nilpotent.

The question of whether the p -torsion-freeness of N in 6.12 can be dropped is a more complex one. As an illustration of the possibilities we prove the following variant of Theorem 6.12.

6.14 THEOREM. *Let p be a prime and G a p -group with a nilpotent normal subgroup N such that G/N is finite. If $\sigma(G : N) = 1$ and G has the subnormal intersection property, G is nilpotent.*

Proof: Denote by P the maximal p -radicable subgroup of N . By Corollary 3.39, P is abelian. Now if $n \in N$, $\langle n \rangle$ is a subnormal p -subgroup of N , and Lemma 3.46 yields $[P, n] = 1$. Thus $P \leq \zeta_1(N)$.

If S is any subgroup in $\Sigma(G : N)$, we have

$$[P, S] = [P, NS] = [P, G].$$

But if we apply Lemma 3.46 to the group SP we deduce that

$$[P, S] \leq S.$$

It then follows, since S was any subgroup in $\Sigma(G : N)$, that

$$[P, G] \leq \sigma(G : N) = 1.$$

But by Theorem 5.33, G/P is necessarily nilpotent. Thus G is

nilpotent, as claimed.

As in 6.12, we cannot omit either the subnormal intersection property or the condition $\sigma(G : N) = 1$ from the statement of the theorem.

(a) Let G be the group of Example 3.43, an extension of a quasi-cyclic 2-group A by a 2-cycle $\langle x \rangle$. By arguments similar to those of Example 5.13 we can show that every proper subnormal subgroup of G is normal and is contained in A . Thus $\sigma(G : A) = G$. Although G satisfies all the other conditions of Theorem 6.14, G is not nilpotent, for $[A, G] = A$.

(b) For each positive integer i , let $A(i)$ denote a cyclic group of order 2^i . Let A be the direct product of the groups $A(i) : i \geq 1$, and define an action of a 2-cycle $\langle x \rangle$ on A by $a^x = a^{-1}$ for each a in A . Let G be the semidirect product of A and $\langle x \rangle$ with this action. For each positive integer j , define subgroups $B(j)$ and $C(j)$ of G by

$$B(j) = \langle A(i) : i \geq j \rangle, \quad C(j) = \langle B(j), x \rangle.$$

Then each $B(j)$ is a normal subgroup of G , and for $j > 1$, $G/B(j)$ is nilpotent of class precisely $j - 1$. Each $C(j)$ is thus subnormal in G , and $C(j)A = G$. Hence

$$\sigma(G : A) \leq \bigcap \{C(j) : j \geq 1\} = \langle x \rangle.$$

If $\langle x \rangle$ were subnormal then by Lemma 2.22, G would be nilpotent, which is clearly not the case. Thus $\sigma(G : A) = 1$, and the only condition of 6.14 that is not fulfilled by G is the subnormal intersection property.

In view of Theorem 6.14, one is tempted to suggest that Theorem 6.12 may hold without the restriction of p -torsion-freeness on N . However any attempt to synthesise 6.12 and 6.14 to yield the stronger

result will need to circumvent the difficulty that the condition $\sigma(G : N) = 1$ need not be inherited by factor groups of G .

6.2 Soluble minimax groups with the subnormal intersection property

In this section we make use of Theorem 6.12 to show that any soluble minimax group with the subnormal intersection property has a bound on its subnormal indices. We first prove a theorem on a very restricted situation.

6.21 THEOREM. *Let G be a soluble minimax group with a nilpotent normal subgroup N such that G/N is finite. Then G has the subnormal intersection property if and only if G has a bound on its subnormal indices.*

Proof: The implication in one direction is immediate. Suppose that counterexamples to the converse implication exist, where by a counterexample we mean an ordered pair (G, N) such that G and N satisfy the postulates of the theorem, G has the subnormal intersection property but G has unbounded subnormal indices. We begin with a useful observation.

(i) *If (G, N) is a counterexample and T is the torsion subgroup of N , then $(G/T, N/T)$ is also a counterexample.*

Proof: All we need to verify is that G/T has no bound on its subnormal indices; but this follows easily from Theorem 4.43, in view of the fact that T , being a periodic soluble minimax group, satisfies the minimal condition on subgroups.

Now it is clear that in any counterexample (G, N) , N is a proper subgroup of G , so the set of positive integers

$$\{|G/N| : (G, N) \text{ is a counterexample}\}$$

has a least element $k > 1$. For brevity we will term any counterexample (G, N) with $|G/N| = k$ a "minimal" counterexample. We note two important facts.

(ii) If (G, N) is a minimal counterexample then N is the maximal nilpotent normal subgroup of G .

This is immediate from our choice of k .

(iii) If (G, N) is a minimal counterexample then $\sigma(G : N) \leq N$ and $(G/\sigma(G : N), N/\sigma(G : N))$ is a minimal counterexample.

Proof: If H is one of the finite set of subnormal subgroups K of G such that $N \leq K < G$, it is clear that (H, N) cannot be a counterexample and H must have a bound on its subnormal indices. Indeed a bound r can be chosen which is independent of H . Now if S is a subnormal subgroup of G such that $SN < G$, it is easy to see that the defect of S in G cannot exceed $r + k$. It follows that the set of non-negative integers

$$\{s(G : S) : S \in \Sigma(G : N)\}$$

must be unbounded, and that therefore $(G/\sigma(G : N), N\sigma(G : N)/\sigma(G : N))$ is a counterexample, indeed a minimal counterexample. Thus $|G/N\sigma(G : N)| = |G/N| = k$, and $\sigma(G : N) \leq N$. This completes the proof of (iii).

We now choose a minimal counterexample (G_1, N_1) in which N_1 is torsion-free; this is possible by (i). Our aim is to obtain a minimal counterexample (G_2, N_2) in which N_2 is torsion-free and $\sigma(G_2 : N_2) = 1$. If $\sigma(G_1 : N_1) = 1$ we need search no further; if not, we put $M_1 = \sigma(G_1 : N_1)$. Then by (iii), $M_1 \leq N_1$ and $(G_1/M_1, N_1/M_1)$ is a minimal counterexample. If we write T_1/M_1 for the torsion subgroup of N_1/M_1 , then $(G_1/T_1, N_1/T_1)$ is also a minimal counterexample. By repeating this process we construct an ascending chain

$$M_1 \leq T_1 \leq M_2 \leq T_2 \leq \dots$$

following generalisation of

of normal subgroups of G_1 , each contained in N_1 , such that for $i \geq 1$, $(G_1/M_i, N_1/M_i)$ and $(G_1/T_i, N_1/T_i)$ are minimal counterexamples, T_i/M_i is the torsion subgroup of N_1/M_i , and $M_{i+1}/T_i = \sigma(G_1/T_i : N_1/T_i)$.

But now N_1 is a soluble group of finite torsion-free rank. Thus by lemma 4.41 there is an integer n such that M_{n+1}/T_n is periodic. Since N_1/T_n is torsion-free, this means that $M_{n+1} = T_n$ or that $\sigma(G_1/T_n : N_1/T_n) = 1$. If we now put $G_2 = G_1/T_n$ and $N_2 = N_1/T_n$, (G_2, N_2) is a minimal counterexample, N_2 is torsion-free and $\sigma(G_2 : N_2) = 1$.

But G_2/N_2 is a non-trivial finite soluble group; we can therefore find a non-trivial minimal normal subgroup Y/N_2 of G_2/N_2 . Then Y/N_2 is an elementary abelian p -group for some prime p , and $\sigma(Y : N_2) = 1$ by remark (b) after Definition 6.11. Moreover Y inherits the subnormal intersection property and N_2 is torsion-free. Thus Y satisfies all the requirements of Theorem 6.12, and so is nilpotent. This contradicts (ii) above and shows that in fact no counterexamples exist, completing the proof of the theorem.

We now state a result of D. McDougall.

6.22 THEOREM (Theorem A of [20]). *Let G be a soluble minimax group. If G has the subnormal intersection property then G is an extension of a radicable abelian group, satisfying the minimal condition on subgroups, by a (torsion-free nilpotent)-by-finite group.*

We can combine this result with Theorem 6.21 to deduce the following generalisation of 6.21.

6.23 THEOREM. *A soluble minimax group has the subnormal intersection property if and only if it has a bound on its subnormal indices.*

Proof: To prove the non-trivial half of the theorem, let G be a soluble minimax group with the subnormal intersection property. By 6.22, G has a normal abelian subgroup A satisfying the minimal condition on subgroups, such that G/A is a finite extension of a torsion-free nilpotent group. By 6.21, G/A has a bound on its subnormal indices, and by Theorem 4.43 so also has G .

6.3 An example

The work of this chapter has shown that under a restriction to soluble minimax groups, the subnormal intersection property is equivalent to the property of having bounded subnormal indices; that is, under this restriction there is equality between the corresponding classes of groups. It would be of interest to know how far this condition can be relaxed before the two classes cease to coincide. In this section we partially answer this question by constructing an example which shows that a restriction to soluble groups of finite reduced rank (in the sense of 4.16 (d)) will not suffice to ensure equality of the two classes.

Our first example will be used as a building block in a more complicated construction.

6.31 EXAMPLE. Let p be any odd prime, and q a prime which divides $p - 1$. Let A be a cyclic group of order p . Since the automorphism group of A has order $p - 1$, A has an automorphism θ of order q . There is a positive integer m such that for any a in A , $a\theta = a^m$; m can clearly be chosen so that $1 < m < p$. Now for each a in A ,

$$\alpha = \alpha\theta^q = \alpha^{m^q},$$

from which it follows that $m^q - 1$ is divisible by p . Let p^α be the largest power of p dividing $m^q - 1$.

Let k be a positive integer, and B a cyclic group of order $p^{k\alpha}$. Since m is coprime to p , the map x defined by $b^x = b^m$, for each b in B , is an automorphism of B . We now prove two useful facts about the action of the automorphism group $\langle x \rangle$ on B .

(i) Let B_1 be any subgroup of B . If r is a positive integer coprime to q then $[B_1, x^r] = B_1$.

Proof: Suppose p divides $m^r - 1$, that is, $m^r \equiv 1 \pmod{p}$. Since q and r are coprime, there exist positive integers s and t such that $sr = 1 + tq$. Then, recalling that $m^q \equiv 1 \pmod{p}$,

$$m^{sr} = m^{1+tq} \equiv m \pmod{p}.$$

But we also have $m^{sr} \equiv 1 \pmod{p}$, yielding $m \equiv 1 \pmod{p}$, in contradiction to our choice of m . Thus $m^r - 1$ and p are coprime. If $B_1 = \langle b_1 \rangle$ then

$$[b_1, x^r] = b_1^{x^r-1} = b_1^{m^r-1}.$$

Hence

$$B_1 = \langle b_1 \rangle = \langle b_1^{m^r-1} \rangle \leq [B_1, x^r].$$

To complete the proof of (i) we need only point out that the reverse inclusion is trivial since each subgroup of B is invariant under the action of $\langle x \rangle$.

(ii) Let B_1 be any subgroup of B . Then $[B_1, x^q] = B_1^{p^\alpha}$.

Proof: Let $B_1 = \langle b_1 \rangle$. Then $[B_1, x^q]$ is generated by the element

$$[b_1, x^q] = b_1^{x^{q-1}} = b_1^{m^{q-1}} = b_1^{p^\alpha n},$$

where, by the choice of α , n and p are coprime. Thus

$$B_1^{p^\alpha} = \langle b_1^{p^\alpha} \rangle = \langle b_1^{p^\alpha n} \rangle = [B_1, x^q],$$

proving (ii).

It is a simple matter to deduce from (ii) that

$$(iii) \quad B^{(x^{q-1})^k} = 1 \quad \text{but} \quad B^{(x^{q-1})^{k-1}} \neq 1.$$

If we now consider the natural semidirect product G of B and $\langle x \rangle$ with the given action, we can describe the subnormal structure of G as follows.

(iv) If a subnormal subgroup S of G contains an element of the form $x^r b$ where r is a positive integer coprime to q , and $b \in B$, then we can apply (i) above, together with the commutativity of B , to deduce that for each non-negative integer i , $B = \gamma B S^i$. Thus $B \leq S$, and S is normal in G .

(v) If a subnormal subgroup S of G does not have the property postulated in (iv), then S lies in the subgroup $\langle B, x^q \rangle$ of G . By (iii), this subgroup is nilpotent of class precisely k , since B is abelian. Thus the defect of S in G cannot exceed $k+1$.

Indeed (iv) and (v) together imply that $k+1$ is a bound for the subnormal indices in G .

We are now in a position to construct the main example of this section.

6.32 EXAMPLE. We first construct a sequence of pairs of primes.

Choose any prime $q(1)$. By Dirichlet's Theorem (Theorem 15 of [13]) the set of positive integers $\{1+kq(1) : k \geq 1\}$ contains infinitely many primes. Let $p(1)$ be one of these primes; then $p(1) - 1$ is divisible by $q(1)$.

Suppose now that we have chosen $(p(1), q(1)), \dots, (p(i), q(i))$ for $i \geq 1$, with the following properties:

- (a) $p(j)$ and $q(j)$ are primes, for $1 \leq j \leq i$;
- (b) $q(j)$ divides $p(j) - 1$, for $1 \leq j \leq i$;
- (c) $q(j) < p(j) < q(m)$, if $1 \leq j < m \leq i$.

Now choose a prime $q(i+1) > p(i)$, and, as above, apply Dirichlet's Theorem to obtain a prime $p(i+1)$ with the property that $q(i+1)$ divides $p(i+1) - 1$. In this way we define recursively a sequence $\{(p(i), q(i)) : i \geq 1\}$ of pairs of primes with the properties (b) and (c) for arbitrary i .

Now, by the discussion in Example 6.31, to each pair $(p(i), q(i))$ there corresponds a positive integer $m(i)$, with $1 < m(i) < p(i)$, such that if $A(i)$ denotes a cyclic group of order $p(i)$ then the map $\theta(i)$, defined by $a\theta(i) = a^{m(i)}$ for each a in $A(i)$, is an automorphism of order $q(i)$. Let $p(i)^{\alpha(i)}$ be the largest power of $p(i)$ dividing $m(i)^{q(i)} - 1$, and let $B(i)$ be a cyclic group of order $p(i)^{\alpha(i)}$. Then, as in 6.31, we can define the action of an infinite cycle $\langle x \rangle$ on $B(i) = \langle b(i) \rangle$ by putting

$$b(i)^x = b(i)^{m(i)}$$

and extending the definition in the obvious way. The properties of this action and of the corresponding semidirect product of $B(i)$ and $\langle x \rangle$ will be precisely as in (i)-(v) of 6.31, with k replaced by i .

If we now denote by B the direct product of the groups $B(i) : i \geq 1$, it is clear that the action of $\langle x \rangle$, defined on each

component $B(i)$ as above, can be extended to B . Consider the natural semidirect product of B and $\langle x \rangle$ with this action.

Firstly we note that since B has p -rank at most 1 for each prime p , B has finite reduced rank in the sense of 4.16 (d). Clearly G is then a metabelian group of finite reduced rank, as defined in 4.31.

Secondly we show that G has the subnormal intersection property. Consider the standard series of an arbitrary subgroup S of G . If S lies in B this series terminates after at most two steps. On the other hand, if $SB > B$, then S must have an element of the form $x^t b$, where $b \in B$ and t is a positive integer. Denote by $\pi(t)$ the set of primes

$$\{q(i) : q(i) \text{ and } t \text{ are coprime}\},$$

a set which clearly contains all but finitely many of the $q(i)$.

Thus if $D(t)$ is the direct product of the subgroups $B(i)$ for which $q(i) \in \pi(t)$, $B/D(t)$ is a finite group and $D(t)$ is normal in G .

It follows from Lemma 2.34 that $G/D(t)$ has a bound on its subnormal indices. But if $q(i) \in \pi(t)$, we have, by (i) of 6.31 and the commutativity of B ,

$$[B(i), S] \geq [B(i), x^t b] = [B(i), x^t] = B(i).$$

It follows that every term of the standard series of S contains $B(i)$. Thus the standard series of S is the same as that of $SD(t)$, hence must terminate after finitely many steps. We have thus verified that G has the subnormal intersection property, by Lemma 2.41.

Now suppose G has a bound y for its subnormal indices. Choose an integer $i > R(y)$, where R is the function of Theorem 2.36. G has a factor group isomorphic to $\langle B(i), x \rangle$. This factor group, $G(i)$ say, inherits the bound y for its subnormal indices. But by (iii) of 6.31, $G(i)$ has a normal subgroup which is nilpotent of

class precisely i and cannot, by Theorem 2.36, have the bound y for its subnormal indices. This contradiction shows that G has unbounded subnormal indices, and completes our investigation of this example.

I have not been able to decide whether soluble groups of finite total rank with the subnormal intersection property need have a bound for their subnormal indices. On the one hand, the fact that factor groups of groups in this class may lie outside the class makes it difficult to handle; on the other hand a counterexample in the spirit of 6.32 is not easy to construct.

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The subnormal structure of metanilpotent groups

D.J. McCaughan and G. McNulty

Let G be a group with a normal subgroup N such that G/N is periodic and N is a p -subgroup of G/N and N is a normal subgroup of G . It is proved that if N is a normal subgroup of G and the subnormal index of N in G is finite, then there is a positive integer k such that every subnormal subgroup of G is nilpotent of class at most k . It is also shown that if G is a periodic group and N is the maximal nilpotent subgroup of G , then N is bound on the subnormal index of N in G if there is a positive integer k such that for all primes p the Sylow p -subgroups of G/N are nilpotent of class at most k .

1. Introduction

A subgroup H of a group G is subnormal in G if H can be connected to G by a finite chain of subgroups each of which is normal in its successor. If such chains exist then there is one of shortest length. The number of strict inclusions in this chain is called the subnormal index (or defect) of H in G . Groups in which every subnormal subgroup has subnormal index at most one are precisely those groups in which conjugacy is transitive, and have been studied in the context of finite groups by Gasf and Tarsky, [1], Gaschütz, [2], and Zacher, [3]. For infinite soluble groups a similar study has been made by Gasf and Tarsky, [4]. Soluble groups in which every subnormal subgroup has subnormal index at

The subnormal structure of metanilpotent groups

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Let G be a group with a normal nilpotent subgroup N such that G/N is periodic and nilpotent. If $G(p)/N$ is the Sylow p -subgroup of G/N and $Q(p)$ is the maximal p -radicable subgroup of N , it is shown that G has a bound on the subnormal indices of its subnormal subgroups if and only if there is a positive integer c such that $G(p)/Q(p)$ is nilpotent of class at most c , for all primes p . It is also shown that if G is a periodic metanilpotent group and Q is its maximal radicable abelian normal subgroup then G has a bound on its subnormal indices if and only if there is a positive integer c such that for all primes p the Sylow p -subgroups of G/Q are nilpotent of class at most c .

1. Introduction

A subgroup H of a group G is subnormal in G if H can be connected to G by a finite chain of subgroups each of which is normal in its successor. If such chains exist then there is one of minimal length; the number of strict inclusions in this chain is called the subnormal index (or defect) of H in G . Groups in which every subnormal subgroup has subnormal index at most one are precisely those groups in which normality is transitive, and have been studied in the context of finite groups by Best and Taussky, [2], Gaschütz, [6], and Zacher, [20]; in the case of infinite soluble groups a similar study has been made by Robinson in [12]. Soluble groups in which every subnormal subgroup has subnormal index at

most n , for an arbitrary positive integer n , have been studied by Robinson, with the added restriction that the groups be standard wreath products of two nilpotent groups, in [16], and by McDougall, with a restriction to p -groups, in [11]. In this paper we are primarily concerned with metanilpotent groups in which every subnormal subgroup has subnormal index at most n , for some positive integer n .

In common with many other investigations of this type, the basis for our results is a lemma dealing with a much simplified situation. However before we can discuss this we need some additional terminology. Let π be a non-empty set of primes. A group G is said to be quasi- π -radicable if, for each π -number k , G is generated by the k -th powers of its elements. A group is π -reduced if it has no non-trivial quasi- π -radicable subgroups. If a group G is an extension of a p -reduced nilpotent group by a cyclic p -group, and has the property that the intersection of any set of subnormal subgroups is again subnormal, then G is nilpotent (Lemma 3.5). In §3 we use this key result to obtain a characterization of nilpotent-by-(periodic nilpotent) groups which have a bound for the subnormal indices of their subnormal subgroups. By Theorem E of [16] the standard unrestricted wreath product of an arbitrary torsion-free abelian group with itself has the property that every subnormal subgroup has subnormal index at most two. If we take the torsion-free group to be infinite cyclic we see that, in contrast to Lemma 3.5, an extension of a reduced (that is, π -reduced where π is the set of all primes) abelian group by a cyclic group can have a bound on its subnormal indices and yet not be nilpotent. Thus a different approach will be needed to deal with arbitrary metanilpotent groups which have a bound on their subnormal indices, and any characterization is likely to be extremely complex.

Results analogous to those of §3 are sought in §4 in connection with groups in which the subnormal subgroups form a complete lattice, since such groups have the property that arbitrary intersections of subnormal subgroups are subnormal and thus Lemma 3.5 is still available. Theorem 4.1 gives necessary conditions for the subnormal subgroups to form a complete lattice, but Theorem 4.3 shows that these conditions are not sufficient. Conversely Theorem 4.4 gives sufficient conditions which, as shown in Theorem 4.6, are not necessary. It seems likely that somewhere between these two sets of conditions lies a set of necessary and sufficient

conditions, but we have been unable to locate it. If G is a periodic metanilpotent group and Q is its maximal radicable abelian normal subgroup, then our characterization can be rephrased in terms of the Sylow p -subgroups of G/Q . Thus in a group in which Q is trivial we have the surprising result that a knowledge of the Sylow p -subgroups suffices for us to decide whether the group has subnormal subgroups of arbitrary subnormal index.

2. Preliminaries

Our aim in this section is to summarise some elementary facts about subnormal subgroups and about quasi- π -radicability. Most of these facts will be used in §3 and §4, but no explicit mention will be made in these later sections.

2.1 Subnormal subgroups. The concept of subnormal subgroup can be approached via the idea of the standard series of a subgroup. The standard series of H in G is the series $\{H_i : i \geq 0\}$, where $H_0 = G$ and H_i is the normal closure of H in H_{i-1} . Then H is subnormal in G if and only if $H_n = H$ for some non-negative integer n . If we let $[H, K]$ denote the commutator of two subgroups H and K in a group, and define γ_{HK}^i inductively by:- $\gamma_{HK}^0 = H$, $\gamma_{HK}^i = [\gamma_{HK}^{i-1}, K]$, then the n -th term of the standard series of H in G can be written as $H\gamma_{GH}^n$. Thus H is subnormal in G if and only if γ_{GH}^i is contained in H for some non-negative integer i . If γ_{GH}^r is contained in H but γ_{GH}^{r-1} is not contained in H then H has subnormal index r in G .

A group G has the subnormal intersection property if the intersection of an arbitrary collection of subnormal subgroups of G is subnormal in G . A group has the subnormal join property if the join of an arbitrary collection of subnormal subgroups is subnormal.

Robinson has proved that every subnormal subgroup of a group G has subnormal index at most n if and only if $H_n = H_{n+1}$ for all subgroups H of G , and that a group has the subnormal intersection property if and only if the standard series of every subgroup becomes stationary after

finitely many terms (Lemma 2 of [14]). He also shows (Lemma 8.1 of [13]) that a group has the subnormal join property if and only if the join of every ascending chain of subnormal subgroups is itself subnormal.

The subnormal join property, the subnormal intersection property, and the property of having bounded subnormal indices, are all inherited by subnormal subgroups and homomorphic images.

A group has the property that its subnormal subgroups form a complete lattice if and only if it has the subnormal join property and the subnormal intersection property. A group which has a bound on its subnormal indices has both these properties and hence its subnormal subgroups form a complete lattice. A fuller treatment of this material, with examples, is given in Chapter 3 of [18].

2.2 Quasi- π -radicability. A group is said to be π -radicable (where π is a non-empty set of primes) if, for every π -number k , each element of the group can be expressed as a k -th power of some element of the group. Clearly π -radicability implies quasi- π -radicability, and Černikov, [4], has shown that for ZA-groups the converse holds. It is easy to see that a group generated by quasi- π -radicable subgroups is itself quasi- π -radicable. This fact, together with Černikov's result, shows that every nilpotent group has a unique maximal π -radicable subgroup, which is therefore characteristic. If π is the set of all primes we use the terms radicable and reduced, instead of π -radicable and π -reduced.

It is easy to prove that an extension of a quasi- π -radicable group by a quasi- π -radicable group is itself quasi- π -radicable. Therefore the factor group of a nilpotent group by its maximal π -radicable subgroup can have no π -radicable non-trivial subgroups, and is therefore π -reduced.

Robinson proves in [15] (Lemma 2.2) that a subnormal periodic radicable abelian subgroup of a group commutes with any subnormal periodic nilpotent subgroup. Therefore in any periodic group G the join of all the subnormal radicable abelian subgroups is the unique maximal radicable abelian normal subgroup of G .

2.3 Notation. We will use $\langle X \rangle$, where X is a set of elements of a group G , to mean the subgroup of G generated by the elements in X . For any positive integer n , the symbol G^n will denote the subgroup of

G generated by the n -th powers of the elements of G .

3. Metanilpotent groups with bounded subnormal indices

In order to obtain the main results of this section we need a series of preliminary lemmas.

LEMMA 3.1. *Let θ be an automorphism of order p^r of an abelian group A , where p is a prime and r is a non-negative integer. Let*

$a\psi = a^{-1}a\theta$ for all elements a of A . Then $A\psi^{p^r} \leq A^p$.

Proof. If we denote the identity endomorphism of A by 1 then in the ring of endomorphisms of A we have $\psi = \theta - 1$. Therefore

$$\psi^{p^r} = (\theta - 1)^{p^r} = \theta^{p^r} + (-1)^{p^r} + pL(\theta),$$

where $L(\theta)$ is some polynomial in θ . Since θ has order p^r we have

$$\psi^{p^r} = 1 + (-1)^{p^r} + pL(\theta).$$

If $p = 2$ this reduces to $\psi^{2^r} = 2(1 + L(\theta))$; and if $p \neq 2$ we have

$$\psi^{p^r} = pL(\theta).$$

Let b be an arbitrary element of $A\psi^{p^r}$. Then $b = a\psi^{p^r}$ for some element a of A . Substituting for ψ^{p^r} we have

$$b = a^2 a^{2L(\theta)} \quad \text{if } p = 2,$$

$$b = a^{pL(\theta)} \quad \text{if } p \neq 2.$$

Hence by putting $c = aa^{L(\theta)}$ ($p = 2$) or $c = a^{L(\theta)}$ ($p \neq 2$) we have, for all primes p , that $b = c^p$. This proves the lemma.

COROLLARY 3.2. *If in the situation of Lemma 3.1 there is a subgroup B of A with $B\psi = B$, then B is p -radicable.*

Proof. Since $B\psi = B$ it follows that $B\theta \leq B$ and hence, since θ has finite order, that $B\theta = B$. Therefore θ restricted to B is an

automorphism of order p^t , where $0 \leq t \leq r$. Hence by the lemma $B\psi^{p^t} \leq B^{p^t}$. However $B\psi^{p^t} = B$, and so B is p -radicable.

LEMMA 3.3. Let $G = \langle x, A \rangle$, where A is a normal abelian subgroup of G . Suppose that $[x^{p^n}, A] = 1$ for some prime p and non-negative integer n . Let $A_0 = A$ and define A_{i+1} inductively by:-
 $A_{i+1} = [A_i, x]$. If $A_k = A_{k+1}$ for some k , and A is p -reduced, then G is nilpotent.

Proof. Conjugation by x gives an automorphism of A whose order divides p^n . Putting $a\psi = a^{-1}a^x$ for all elements a of A we have $A_k\psi = A_{k+1} = A_k$. Therefore by Corollary 3.2, A_k is p -radicable. But A is p -reduced and so A_k is trivial. Therefore A lies in the k -th term of the upper central series of G , and so G is nilpotent.

LEMMA 3.4. If π is a non-empty set of primes and N is a nilpotent π -reduced group, then N/Z is π -reduced, where Z is the centre of N .

Proof. If N is abelian there is nothing to prove, so we may assume N is not abelian. Let H/Z be the maximal π -radicable subgroup of N/Z . Then H is a normal subgroup of N and is π -reduced. We will prove that H is abelian. Suppose on the contrary that H is not abelian. Then there is an element y of H which is in the second term of the upper central series of H but is not central. Let A be the centre of H . We define a homomorphism θ of H/A into H by:-

$$(hA)\theta = [h, y] \text{ for all } h \text{ in } H.$$

Since A contains Z , the image of H/A under θ is π -radicable, and so, since H is π -reduced, must be trivial. Thus y commutes with all h in H and is therefore central. This is a contradiction and so H is abelian.

Since H is abelian it is a proper subgroup of N . Let x be an element of N not in H . We define a map ψ of H into H by:-

$$h\psi = [h, x] \text{ for all } h \text{ in } H.$$

Since H is abelian and normal in N , ψ is a homomorphism of H into H . The kernel of ψ contains Z and so $H\psi$ is π -radicable. Since N is π -reduced we have $[h, x] = 1$ for all h in H and all x in N , so that H is contained in the centre of N . Therefore $H = Z$ and N/Z is π -reduced.

LEMMA 3.5. *Let $G = \langle x, N \rangle$ where N is a p -reduced normal nilpotent subgroup of G and G/N has order p^n . If G has the subnormal intersection property then G is nilpotent.*

Proof. We use induction on the nilpotent class of N . If this class is zero, that is if N is trivial, then the result is clearly true. Let A be the centre of N . By Lemma 3.4, N/A is p -reduced and so by the induction hypothesis G/A is nilpotent. Therefore $\langle x, A \rangle/A$ is subnormal in G/A and hence $\langle x, A \rangle$ has the subnormal intersection property. Since x^{p^n} lies in N we have $[x^{p^n}, A] = 1$. Let $M = \langle x \rangle \cap A$. Then M is a central subgroup of $\langle x, A \rangle$. The standard series of $\langle xM \rangle$ in $\langle x, A \rangle/M$ is easily seen to be $\{\langle x \rangle A_i/M : i \geq 0\}$, where A_i is defined as in Lemma 3.3 ($A_0 = A$, $A_{i+1} = [A_i, x]$). Since $\langle x, A \rangle/M$ has the subnormal intersection property, $\langle x \rangle A_t/M = \langle x \rangle A_{t+1}/M$ for some t . Therefore A_t^M/M is contained in $\langle xM \rangle A_{t+1}^M/M$, and since this latter group is a split extension it follows that $A_t^M = A_{t+1}^M$. Therefore

$$A_{t+1} = [A_t, x] = [A_t^M, x] = [A_{t+1}^M, x] = [A_{t+1}, x] = A_{t+2}.$$

Thus we can apply Lemma 3.3 to deduce that $\langle x, A \rangle$ is nilpotent. G is therefore the join of a normal nilpotent subgroup N and a subnormal nilpotent subgroup $\langle x, A \rangle$, and hence must be nilpotent (Lemma 4.5 of [13]).

LEMMA 3.6. *Let N be a nilpotent group with upper central series $\{Z_i : 0 \leq i \leq c\}$. Let π be a non-empty set of primes. Define R_i inductively by:- $R_0 = 1$, R_i/R_{i-1} is the maximal π -radicable subgroup of the centre of N/R_{i-1} . Then R_i is the maximal π -radicable subgroup of Z_i and in particular $R_c = R_{c+1}$.*

Proof. We use induction on i . The case $i = 0$ is trivial. Suppose R_{i-1} is the maximal π -radicable subgroup of Z_{i-1} . By definition $[R_i, N] \leq R_{i-1}$, and since R_{i-1} is a subgroup of Z_{i-1} it follows that R_i is a subgroup of Z_i . Since R_i is π -radicable it remains to prove that it is the maximal π -radicable subgroup of Z_i . Let R be the maximal π -radicable subgroup of Z_i . Then the commutator of R and N is contained in Z_{i-1} .

Let us assume that R/R_{i-1} is abelian. For fixed g in N the mapping τ_g which sends rR_{i-1} to $[r, g]R_{i-1}$, for all r in R , is a homomorphism of R/R_{i-1} into Z_{i-1}/R_{i-1} . But R/R_{i-1} is π -radicable and Z_{i-1}/R_{i-1} is π -reduced, and so the kernel of τ_g must be the whole of R/R_{i-1} . It follows that R/R_{i-1} is contained in the centre of N/R_{i-1} . However R_i/R_{i-1} is the maximal π -radicable subgroup of the centre of N/R_{i-1} and so $R = R_i$. Thus it remains to show that R/R_{i-1} is indeed abelian.

π -radicability is a tensorial property (in the sense of [17]) and so is inherited by each factor of the lower central series of R (Theorem 1 of [17]). Since R is nilpotent it follows that R' , the derived group of R , is π -radicable. But since R' is contained in Z_{i-1} it follows that R' is in fact contained in R_{i-1} . Thus R/R_{i-1} is abelian and the proof of the lemma is complete.

LEMMA 3.7. *Let N be a nilpotent normal subgroup of a group G , with G/N a π -group for some non-empty set of primes π . Let $Q(\pi)$ be the maximal π -radicable subgroup of N . If N has nilpotent class c and S is any subnormal subgroup of G then the subnormal index of S in $SQ(\pi)$ is at most c .*

Proof. Let $R_0, R_1, R_2, \dots, R_c$ be chosen as in Lemma 3.6. Since each R_i is normal in G the groups SR_i are subnormal in G and, since R_{i+1} normalises $SR_i \cap N$, it follows that $SR_i/SR_i \cap N$ is a subnormal

subgroup of $SR_{i+1}/SR_i \cap N$. However $SR_i/SR_i \cap N$ is a π -group and $R_{i+1}(SR_i \cap N)/SR_i \cap N$ is π -radicable and abelian. Therefore by Lemma 4 of [16], $SR_i/SR_i \cap N$ is normal in $SR_{i+1}/SR_i \cap N$. Hence SR_i is normal in SR_{i+1} for $0 \leq i \leq c-1$. By Lemma 3.6, $R_c = Q(\pi)$ and so the subnormal index of S in $SQ(\pi)$ is at most c .

We now have enough lemmas at our disposal to make the proof of the main theorem relatively short.

THEOREM A. *Let N be a normal nilpotent subgroup of a group G such that G/N is periodic and nilpotent. For any prime p let $Q(p)$ be the maximal p -radicable subgroup of N and let $G(p)/N$ be the Sylow p -subgroup of G/N . Then G has a bound on the subnormal indices of its subnormal subgroups if and only if there is a positive integer c such that $G(p)/Q(p)$ is nilpotent of class at most c , for all primes p .*

Proof. Suppose that every subnormal subgroup of G has subnormal index at most n . Then each group $G(p)/Q(p)$ has the same property. $N/Q(p)$ is p -reduced and $G(p)/N$ is a nilpotent p -group. Therefore, if x is any element of $G(p)$, the group $\langle x, N \rangle/Q(p)$ has the subnormal intersection property, and so by Lemma 3.5, $\langle x, N \rangle/Q(p)$ is nilpotent. It follows that $\langle x, Q(p) \rangle/Q(p)$ is subnormal in $G(p)/Q(p)$. This means that every cyclic subgroup of $G(p)/Q(p)$ is subnormal. Since $G(p)/Q(p)$ also has the subnormal join property it follows that every subgroup of $G(p)/Q(p)$ is subnormal. Hence in $G(p)/Q(p)$ every subgroup is subnormal with subnormal index at most n , and so by a result of Roseblade (Corollary to Theorem 1 of [19]), $G(p)/Q(p)$ is nilpotent of class at most $f(n)$. Putting $c = f(n)$ we have one half of the theorem.

Conversely, suppose that there is a positive integer c such that $G(p)/Q(p)$ is nilpotent of class at most c for all primes p . Let the nilpotent class of N be d . We will show that if S is subnormal in G then the subnormal index of S in G is at most $(d+1)c + \frac{1}{2}d(d+3)$. Let us denote this expression by $f(d)$. We will proceed by induction on d . If $d = 0$ then G is nilpotent of class at most c and the result is trivial. Let Z be the centre of N . Then $Q(p)Z/Z$ is contained in the maximal p -radicable subgroup of N/Z . Since $G(p)/Q(p)Z$ is nilpotent of class at most c it follows that the hypotheses are satisfied by G/Z .

Therefore by the induction hypothesis every subnormal subgroup of G/Z has subnormal index at most $f(d-1)$. In particular if S is subnormal in G then SZ/Z has subnormal index at most $f(d-1)$ in G/Z . Thus SZ has subnormal index at most $f(d-1)$ in G . It remains to find a bound on the subnormal index of S in SZ .

$S/S \cap N$ is a subnormal subgroup of $SZ/S \cap N$. Since $S/S \cap N$ is periodic and nilpotent we can write S as the product of its normal subgroups S_p , where $S_p/S \cap N$ is the Sylow p -subgroup of $S/S \cap N$. Since $S_p N/N$ is isomorphic to $S_p/S_p \cap N$, which is a p -group, S_p is contained in $G(p)$. However $G(p)/N$ is a p -group and $Q(p)$ is the maximal p -radicable subgroup of N , and so by Lemma 3.7 the subnormal index of S_p in $S_p Q(p)$ is at most d . But $G(p)$ is normal in G and $G(p)/Q(p)$ is nilpotent of class at most c . Therefore the subnormal index of S_p in G is at most $c + d + 1$. Thus the subnormal index of $S_p/S \cap N$ in $SZ/S \cap N$ is at most $c + d + 1$. Since $SZ/S \cap N$ is the join of a nilpotent normal subgroup $Z(S \cap N)/S \cap N$ and a subnormal nilpotent subgroup $S/S \cap N$ it must be nilpotent (Lemma 4.5 of [13]). Hence the terms of the standard series of $S/S \cap N$ in $SZ/S \cap N$ all lie in the torsion-subgroup of $SZ/S \cap N$ (apart from the 0-th term, which is $SZ/S \cap N$ itself), and so each term is the product of the corresponding terms of the standard series of the subgroups $S_p/S \cap N$. Therefore the subnormal index of $S/S \cap N$ in $SZ/S \cap N$ is at most $c + d + 1$.

Putting the two pieces together it follows that the subnormal index of S in G is at most $f(d-1) + c + d + 1$. But

$$f(d-1) = dc + \frac{1}{2}(d-1)(d+2) = dc + \frac{1}{2}(d^2+d-2) .$$

Therefore

$$\begin{aligned} f(d-1) + c + d + 1 &= (d+1)c + \frac{1}{2}(d^2+d-2+2d+2) \\ &= (d+1)c + \frac{1}{2}d(d+3) \\ &= f(d) . \end{aligned}$$

Thus the subnormal index of S in G is at most $f(d)$.

COROLLARY. *Let N be a nilpotent normal subgroup of a group G , with G/N periodic and nilpotent. Let $Q(p)$ be the maximal p -radicable*

subgroup of N and let $G(p)/N$ be the Sylow p -subgroup of G/N . If there is a positive integer c such that $G(p)/Q(p)$ is nilpotent of class at most c , for all primes p , then the subnormal index in G of any subnormal subgroup is at most $f(d)$, where d is the nilpotent class of N and $f(d) = (d+1)c + \frac{1}{2}d(d+3)$.

For periodic metanilpotent groups the following characterization is probably more useful, since it involves Sylow subgroups.

THEOREM B. Let G be a periodic metanilpotent group, and let Q be the maximal radicable abelian normal subgroup of G . Then G has a bound on its subnormal indices if and only if there is a positive integer c such that for all primes p the Sylow p -subgroups of G/Q are nilpotent of class at most c .

Proof. There is a normal nilpotent subgroup B of G such that G/B is nilpotent. By Fitting's Theorem, BQ is nilpotent. Thus if we put $N = BQ$ we have that N is a normal nilpotent subgroup of G and G/N is nilpotent. The result will follow from Theorem A if we can show that, for any prime p , the Sylow p -subgroups of G/Q are nilpotent of class at most c if and only if $G(p)/Q(p)$ is nilpotent of class at most c (where $G(p)$ and $Q(p)$ are as defined in Theorem A). $Q(p)$, since it is periodic and nilpotent, is the direct product of a p' -group (where p' denotes the set of primes other than p) with a radicable p -group P . But by a result of Černikov, [3], a radicable nilpotent p -group must be abelian, and so P is contained in Q . Therefore $Q(p)/Q$ is a p' -group. Also, since $Q(p)$ contains all the p' -elements of N , $G(p)/Q(p)$ is a p -group.

Let $G(p)/Q(p)$ be nilpotent of class at most c , and let S/Q be a Sylow p -subgroup of G/Q . Since $S/Q \cap Q(p)/Q = 1$ we know that $S \cap Q(p) = Q$. But $SQ(p)/Q(p)$ is isomorphic to $S/S \cap Q(p) = S/Q$, and since $SQ(p)/Q(p)$ is a subgroup of $G(p)/Q(p)$ it follows that S/Q is nilpotent of class at most c .

Conversely, suppose that the Sylow p -subgroups of G/Q are nilpotent of class at most c . Let $\{x_1, \dots, x_r\}$ be any finite set of elements of $G(p)$. Since $G(p)/Q(p)$ is a p -group it follows that $\langle x_1, \dots, x_r \rangle Q(p)/Q(p)$ is a finite p -group. Therefore

$\langle x_1, \dots, x_r \rangle Q(p)/Q$ splits over $Q(p)/Q$, since the latter is a p' -group (Theorem 3 of [5]). Thus we have $\langle x_1, \dots, x_r \rangle Q(p) = S_r Q(p)$, where $S_r \cap Q(p) = Q$. Then S_r/Q is a finite p -group and so by assumption has nilpotent class at most c . Therefore every finitely generated subgroup of $G(p)/Q(p)$ is nilpotent of class at most c , and so $G(p)/Q(p)$ is nilpotent of class at most c . This completes the proof of the theorem.

As an easy application of Theorem B and the corollary to Theorem A we have:-

COROLLARY. *Let G be a quasi-radicable metabelian group satisfying the minimal condition for normal subgroups. Then every subnormal subgroup of G has subnormal index at most four.*

Proof. By a result of Baer, [1], soluble groups satisfying the minimal condition for normal subgroups are periodic. By Corollary 3.3 of [10] the Sylow p -subgroups of G are abelian for all primes p . Thus we have the situation of Theorem B with $c = 1$. Combining Theorem B with the corollary to Theorem A we have that every subnormal subgroup of G has subnormal index at most $f(d)$. Since $d = 1$ and $c = 1$ we find by substitution in the formula that $f(d)$ is then four.

4. Groups in which the subnormal subgroups form a complete lattice

In this section we attempt to find analogues of the previous theorems for the class of groups in which the subnormal subgroups form a complete lattice. A group in this class will have the subnormal intersection property and so Lemma 3.5 can be used.

THEOREM 4.1. *Let N be a nilpotent normal subgroup of a group G , with G/N periodic and nilpotent. For each prime p let $G(p)/N$ be the Sylow p -subgroup of G/N and let $Q(p)$ be the maximal p -radicable subgroup of N . If the subnormal subgroups of G form a complete lattice then $G(p)/Q(p)$ has the property that every subgroup is subnormal, for all primes p .*

Proof. Since the subnormal intersection property is inherited by subnormal subgroups and homomorphic images, $G(p)/Q(p)$ has the subnormal intersection property. Thus for any element x of $G(p)$ we can apply

Lemma 3.5 to $\langle x, N \rangle / Q(p)$. It follows that $\langle x, Q(p) \rangle / Q(p)$ is subnormal in $G(p) / Q(p)$. Since $G(p) / Q(p)$ also has the subnormal join property, every subgroup of $G(p) / Q(p)$, being the join of its cyclic subgroups, must be subnormal.

As in §3 we can restate this result for periodic groups.

THEOREM 4.2. *Let G be a periodic metanilpotent group and let Q be the maximal radicable abelian normal subgroup of G . If the subnormal subgroups of G form a complete lattice then in every Sylow p -subgroup of G/Q each subgroup is subnormal, for all primes p .*

The proof is similar to the corresponding part of the proof of Theorem B, and so will be omitted.

We now give an example to show that the converse to Theorem 4.2 is false.

THEOREM 4.3. *There is a periodic metabelian group which is locally nilpotent but has neither the subnormal intersection property nor the subnormal join property, although in each Sylow p -subgroup every subgroup is subnormal.*

Proof. Let H_p denote the p -group constructed by Heineken and Mohamed in [8]. Then H_p is a metabelian group in which every subgroup is subnormal, and in which the set of subnormal indices is unbounded. Let H be the direct product over all primes p of the groups H_p . Let S_p be a subnormal subgroup of H_p of subnormal index p , and let T_p be the direct product of S_p with all H_q , where $q \neq p$. Then S_p and T_p are subnormal of index p in H . Let T be the join of the subgroups S_p (actually it is just their direct product). It is also the intersection of the subgroups T_p . If T were subnormal of index n in H then each S_p would be subnormal of index at most n in H_p , contrary to the construction for $p > n$. Hence T is not subnormal in H , and so H does not have either the subnormal intersection property or the subnormal join property.

It follows from Theorem 4.3 that the condition that every subgroup of

$G(p)/Q(p)$ be subnormal, for all primes p , is not sufficient to guarantee that the subnormal subgroups of G form a complete lattice. Thus we have a necessary condition which is not sufficient. We next exhibit a set of sufficient conditions which, however, are not necessary.

THEOREM 4.4. *Let N be a normal nilpotent subgroup of a group G with G/N periodic and nilpotent. Let $G(p)/N$ be the Sylow p -subgroup of G/N and let $Q(p)$ be the maximal p -radicable subgroup of N . If there is a positive integer c such that for almost all primes p , $G(p)/Q(p)$ is nilpotent of class at most c , and if for the finitely many exceptional primes p every subgroup of $G(p)/Q(p)$ is subnormal, then the subnormal subgroups of G form a complete lattice.*

Proof. By the remarks made in §2 it will suffice to show that if S is the union of an ascending chain $\{S_i; i < \rho\}$, where ρ is a limit ordinal, of subnormal subgroups of G , and if T is the intersection of an arbitrary family $\{T_i; i \in I\}$ of subnormal subgroups of G , then S and T are again subnormal in G .

We proceed by induction on the class of N . If N has class zero then $N = 1$ and the result is trivial. Let Z be the centre of N . For each prime p , $Q(p)Z/Z$ is a p -radicable subgroup of N/Z , and it follows easily that G/Z inherits the properties of G . Therefore by the induction hypothesis we may assume that G/Z satisfies the conclusion of the theorem. In particular, since $SZ = \bigcup_{i < \rho} S_i Z$, it follows that SZ is

subnormal in G . Thus we need only show that S is subnormal in SZ or, equivalently, that $S/S \cap N$ is subnormal in $SZ/S \cap N$. Each of the groups $S_i Z/S_i \cap N$ is nilpotent, since it is the join of a subnormal nilpotent group $S_i/S_i \cap N$ and a normal abelian subgroup $Z(S_i \cap N)/S_i \cap N$ (Lemma 4.5 of [13]).

Hence $SZ/S \cap N$ is locally nilpotent, since it is the union of the subgroups $S_i Z(S \cap N)/S \cap N$. Since $S/S \cap N$ is periodic it is contained in the torsion-subgroup of $SZ/S \cap N$. Let $S(p)/S \cap N$ be the Sylow p -subgroup of $S/S \cap N$. Then if $S(p, i)/S_i \cap N$ is the Sylow p -subgroup of $S_i/S_i \cap N$ it follows easily that $S(p)$ is the union of the $S(p, i)$. If the nilpotent class of N is d then application of Lemma 3.7 to $S(p, i)N$ shows that $S(p, i)$ has subnormal index at most d in $S(p, i)Q(p)$. Hence $S(p)$

has subnormal index at most d in $S(p)Q(p)$. Let π be the set of primes for which $G(p)/Q(p)$ is nilpotent of class at most c , so that π' is a finite set (π' is the complement of π in the set of all primes). Then if $p \in \pi$ it follows that $S(p)$ is subnormal in G and has subnormal index at most $c + d + 1$. If $p \in \pi'$ then every subgroup of $G(p)/Q(p)$ is subnormal and so $S(p)$ is subnormal in G . As there are only finitely many primes in π' we can find an integer r such that the subnormal index of $S(p)$ in G is at most r , for all primes p . Thus $S(p)/S \cap N$ has subnormal index at most r in $SZ/S \cap N$. Since $SZ/S \cap N$ is locally nilpotent it follows that the n -th term of the standard series of $S/S \cap N$ in $SZ/S \cap N$ is the direct product of the n -th terms of the standard series of the $S(p)/S \cap N$, for $n \geq 1$. Therefore $S/S \cap N$ is subnormal in $SZ/S \cap N$ with subnormal index at most r .

Similarly we can consider $T/T \cap N$. Let $T(p)/T \cap N$ be the Sylow p -subgroup of $T/T \cap N$. If $T(p, i)/T_i \cap N$ is the Sylow p -subgroup of $T_i/T_i \cap N$ then $T(p) = \bigcap_i T(p, i)$. By an argument similar to that above we can prove that each $T(p)$ is subnormal in G . Since T is the join of all the groups $T(p)$ it follows from the first part of the proof that T is subnormal in G . Thus the proof is complete.

As before we can try to rewrite the theorem for periodic groups in terms of Sylow p -subgroups. However we need another assumption, namely that G/N is countable.

THEOREM 4.5. *Let G be a periodic group with a normal nilpotent subgroup N such that G/N is a countable nilpotent group. Let Q be the maximal radicable subgroup of N . Let π be a set of primes such that its complement π' is finite. If there is a positive integer c such that the Sylow p -subgroups of G/Q are nilpotent of class at most c for all p in π , and if every subgroup of each Sylow p -subgroup of G/Q is subnormal for all p in π' , then the subnormal subgroups of G form a complete lattice.*

Proof. It is sufficient to show that the hypotheses of the theorem imply those of Theorem 4.4.

Let $Q(p)$ be the maximal p -radicable subgroup of N , and let $G(p)/N$ be the Sylow p -subgroup of G/N . Since Q is contained in each

$Q(p)$ we can assume $Q = 1$. Therefore $Q(p)$ is the p' -subgroup of N and $G(p)/Q(p)$ is a p -group. The centraliser in $G(p)$ of $Q(p)$ contains the Sylow p -subgroup of N , and so the product of $Q(p)$ and its centraliser in $G(p)$ contains N . Therefore by Theorem 3 of [5], since G/N is countable, $G(p) = SQ(p)$, where $S \cap Q(p) = 1$. Thus S is a p -group and hence is contained in a Sylow p -subgroup of G . If $p \in \pi$ then S is nilpotent of class at most c , so that $G(p)/Q(p)$ is nilpotent of class at most c . If $p \in \pi'$ then every subgroup of $G(p)/Q(p)$ is subnormal. Hence the result.

REMARK. In the previous theorem it would suffice to have $G(p)/N$ countable for all p in π' . The cases where p is in π could be dealt with as in the proof of Theorem B.

Theorem 4.5 gives sufficient conditions for the subnormal subgroups of a countable periodic metanilpotent group to form a complete lattice. We conclude with an example to show that these conditions are not necessary.

THEOREM 4.6. *There is a countable periodic metabelian group which has a Sylow p -subgroup of nilpotent class p for infinitely many primes p , but in which the subnormal subgroups form a complete lattice.*

Proof. Let p be any prime. We put $p(0) = p$ and construct an infinite sequence of primes as follows. Suppose $p(0), p(1), \dots, p(n)$ are already chosen. By Dirichlet's Theorem (Theorem 15 of Hardy and Wright, [7]) the sequence $\{1 + mp(0)p(1) \dots p(n)\}$ where m varies over the positive integers, contains infinitely many primes. We choose $p(n+1)$ to be one such prime. Then $p(n+1) > p(n)$ and $p(n+1)$ is congruent to 1 modulo $p(m)$ for all $m \leq n$.

If $m < n$ then the cyclic group of order $p(n)$ has an automorphism of order $p(m)$. This automorphism takes a generator to a power $\theta(n, m)$ of itself. Let X be the direct product of groups X_i , where X_i is a cyclic group of order $p(i)$ generated by an element $x(i)$, for all non-negative integers i . Let A_i be the direct product of $p(i)$ copies of the cyclic group of order $p(i)$. Let A be the direct product of the groups A_i taken over all non-negative integers i . We show that X can be considered as a group of automorphisms of A . We define an action of

X on A_i as follows:-

let X_i act on A_i via the right regular representation;

let X_j act trivially on A_i for $j > i$;

for all α in A_i and all $j < i$ let $\alpha^{x(j)} = \alpha^{\theta(i,j)}$.

It is easy to see that this makes X into a group of automorphisms of A . Thus we can form the natural split extension $G = AX$, where A is normal in G and $A \cap X = 1$. Clearly G is a countable periodic metabelian group.

Each subgroup $A_i X_i$ is a Sylow $p(i)$ -subgroup of G . But $A_i X_i$ is isomorphic to the standard wreath product of the cyclic group of order $p(i)$ with itself, and so by a result of Liebeck (Theorem 5.1 of [9]) is nilpotent of class $p(i)$. It remains to show that the subnormal subgroups of G form a complete lattice. Let S be a subnormal subgroup of G . Suppose SA contains $x(j)$ for some j . Then $[A, x(j)]$ is generated by the elements $\alpha^{-1} \alpha^{x(j)}$ for all α in A . Taking α in A_i , where $i > j$, then $\alpha^{x(j)} = \alpha^{\theta(i,j)}$ and so $\alpha^{-1} \alpha^{x(j)} = \alpha^{\theta(i,j)-1}$. $\alpha^{\theta(i,j)-1}$ generates the same cyclic subgroup as α . Therefore $[A, x(j)]$ contains A_i for all $i > j$. Thus $\gamma A \langle x(j) \rangle^t$ contains A_i for all $i > j$ and all $t > 0$. Also, since A is abelian, $\gamma AS^t = \gamma A(SA)^t$. Therefore γAS^t contains A_i for all $i > j$, and since γAS^t is contained in S for some t it follows that S contains A_i for all $i > j$.

Let T be the union of an ascending chain $\{S_k; k < \rho\}$, for some limit ordinal ρ , of subnormal subgroups of G . If T is contained in A then it is subnormal in G as required. If not, then for some k the group $S_k A$ is strictly larger than A , and so contains some $x(j)$. Thus by the above remarks S_k contains A_i for all $i > j$. The product of all the A_i with $i > j$ is a normal subgroup $A(j)$ of G . It is easily seen that $G/A(j)$ satisfies the conditions of Theorem 4.5 and so has the

subnormal join property. Since $S/A(j)$ is the union of the $S_\lambda/A(j)$, $\lambda \geq k$, it follows that $S/A(j)$ is subnormal in $G/A(j)$, and hence that G has the subnormal join property.

If S is the intersection of subnormal subgroups S_k of G ($k \in L$) then either S is contained in A and so is subnormal in G , or A is a proper subgroup of SA . If the latter is the case then SA contains $x(j)$ for some j . Therefore $S_k A$ contains $x(j)$ for all k , and so as above S_k contains A_i for all $i > j$ and for all k . Consequently S contains A_i for all $i > j$, and as before we may pass to the factor group $G/A(j)$. But $G/A(j)$ satisfies the hypotheses of Theorem 4.5 and so S is subnormal in G . Therefore G has the subnormal intersection property, and the proof is complete.

REMARK. If we let K be the direct product of the groups $A_i X_i$ for all non-negative i , we can show as in Theorem 4.3 that K has neither the subnormal intersection property nor the subnormal join property. Thus it seems likely that to get necessary and sufficient conditions for the subnormal subgroups to form a complete lattice we need to look somewhere other than just at the groups $G(p)/Q(p)$. In other words a characterization in terms of the $G(p)/Q(p)$ alone seems unlikely in the light of our examples.

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